How well do criterion-optimizing routines for factor rotation in fact optimize?

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ABSTRACT

In Rozeboom, 1993, I argued--primarily on the basis of experience with Hyball rotation--that due to the problem of local optima in nonlinear optimization, analytic factor routines cannot generally be trusted to converge to the axis positions that globally optimize their criterion measure. Here, I present an adjudication of this matter, using simulation data with complex source structures, for the major variants of Orthomax, direct Oblimin, and Hyball. Sensitivity to start position--or, for Orthomax, its lack--is well documented; but more importantly, present results include extensive appraisal of these rotation method's comparative success in source recovery at varied grades of problem difficulty, leading to some unequivocal recommendations on what to use when strong hyperplanes are the goal of rotation.

Key words. Factor rotation; Orthomax; Oblimin; Hyball; Spin search.

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Preface

When devising analytic factor-rotation procedures, it is one thing to define a criterion measure whose optimization defines the target of axis positioning, and something else again to program routines that do in fact locate these optima for inputs to which the algorithm is applied. In Rozeboom, 1993, I reported discovery that the output of my Hyball rotation program (Rozeboom, 1991a, 1991b) is significantly dependent on the initial axis positioning from which the algorithm iterates successive improvements, and argued that the same should be expected from any rotation procedure whose solution is a convergence of criterion-guided iterations. But my description there of an effective solution to this problem was accompanied by little hard data on Hyball's sensitivity to start position and none at all for other rotation methods. I have now completed an extensive simulation study of this matter for three major families of analytic rotation, namely, Orthomax/Promax, direct Oblimin and, of course, Hyball. Its findings are rather instructive.

The results reported here commence with tests of a spectrum of variants within each of the just-named rotation families for their success at source recovery when rotating extraction axes from a standard start position. The best-of-breed then advance to a multifaceted examination of their performance under Spin search, which is the technique described in Rozeboom, 1993, for prevailing over the start problem. The most important facet thereof is appraisal of these methods' respective source-recovery success when Spin-search enables full realization of their criterion's potential for diagnosing best pattern. A second is establishing the extent to which the yield of each method is indeed affected by variation in start position. A third that turns out to be minor, but might well have proved otherwise, compares Serial vs. Parallel concatenation of planar rotations within the solution iteration. A fourth reveals how results for each of the preceding facets vary as a function of noise level in the source-patterns' hyperplanes, that is, the degree of blur in the distinction between salient and non-salient source loadings. Finally, there are provocative findings to discuss on the difference between each method's Spin-search solution judged best by the method's optimization criterion and the one that in fact most closely matches the source structure.

Setup

In order for the results tabled below to be meaningful, I must first clarify this study's technicalities. Specifically, you need details on:

- Composition of the simulation data on which success at source recovery has been appraised.
- Computer implementation of the three tested rotation species.
- The contrast between Serial and Parallel iterated rotation.
- The nature of Spin search.
- The Divergence measure of pattern similarity.

The simulation data.

The test problems for this study were 100 standardized covariance matrices simulating the correlations among *NV* = 25 data variables (items) having *NF* = 5 common factors. These were generated by randomization within frame constraints by the production process described in Rozeboom, 199?. This procedure starts by creating an *NV*-by-*NF* pattern template in which selected elements are tagged as "salient." In the present study, as in Rozeboom 199?, the salient elements comprised all the 25 different ways to realize factor complexities 1, 2, or 3 on five factors. Thus, five items had just one salient factor loading, 10 had two salients, and 10 had three; while 11 of the 25 loadings on each factor were salient. Next, each of 100 raw source patterns was constructed by assigning random numerical values to the template's salient elements with uniform probability in size interval [.25, 1.0] and sign made negative with one of the four probabilities *P* = 0, .10, .20, .30. And nonsalient loadings were assigned with uniform random probability between -*W* and +*W* for one of five hyperplane-noise levels *W* = 0, .05, .10, .15, .20. Each of the 4×5 = 20 fully-crossed <*P*,*W*> combinations was randomly realized five times, yielding a total of 100 raw source patterns. (Differences in *P* had no discernable effect on source recovery by any of the methods tested, and will hereafter be ignored. But as you might expect, *W* proves to be important.) Next, for each of the 100 raw patterns (**R**), the to-be-generated dataset based on **R** was assigned random item communalities in the interval [.25, .80] together with a matrix C_{FF} of semi-random factor correlations, enabling the refined source pattern **A** for this dataset to be derived by a ~~columnwise~~ row-wise rescaling $\mathbf{A} = \mathbf{DR}$ of **R** that put the assigned communalities on the diagonal of common-parts covariance matrix $C_{HH} = \mathbf{A}C_{FF}\mathbf{A}' = \mathbf{DR}C_{FF}\mathbf{R}'\mathbf{D}$.

Production of C_{FF} made 8 of its 10 between-factor correlations nonzero by a collinearity-shielded randomization eventuating in correlation highs averaging .52 over all the datasets, and lows averaging –.32. For details of this procedure, see Rozeboom, 199?

From there, a simulation population P of 5,000 datascore records with composition $Y = \mathbf{A}F + U$ was generated in such fashion that in this P, scores on unique variables $U = \langle u_1,\ldots,u_{25}\rangle$ were precisely orthogonal to themselves and to scores on the simulated source factors $F = \langle f_1,\ldots,f_5\rangle$, scores on F had precisely the assigned correlations C_{FF}; and the score distributions on F and U were approximately Normal with $\mathrm{Diag}[C_{UU}] = \mathbf{I} - \mathrm{Diag}[C_{HH}]$. Thus classical factor model $C_{YY} = \mathbf{A}C_{FF}\mathbf{A}' + C_{UU}$, with C_{UU} diagonal and item variances in C_{YY} all unity, fitted P exactly. Finally, a subject sample of size $NS = 400$ was randomly selected from P, the within-sample correlations among the 25 Y-variables were solved for five principal factors (iterated communalities), and this extraction pattern was archived along with the population's source pattern **A** and covariances C_{FF} awaiting source-recovery tests.

Sample size $NS = 400$ was chosen in light of evidence (Rozeboom, 199?) that sampling disturbance in the best possible solution for source pattern from item correlations at this sampling level is modest but by no means negligible. In simulation studies, this limit on recovery accuracy can be identified rather precisely by noting that any extraction pattern's rotation most closely matched to source should differ at most trivially from the axes obtained by its procrustes rotation to the source pattern as target. In the present study, RMS pattern difference, mean congruence Divergence, and RMS factor-correlation difference between an extraction pattern's procrustes rotation and its source target averaged .050, 7.87°, and .047 respectively over all 100 datasets. (See below for clarification of these measures.) These recovery limits were stunningly unaffected by hyperplane noise: The largest group-mean deviations from these grand means at any W-level were .002, 0.16, and .008 respectively.

The distributions of factor loadings that resulted from this production process, both in populations and in samples, are detailed in Appendix C.

The rotation programs.

As announced above, the three species of factor rotation appraised here are Orthomax/Promax, direct Oblimin, and Hyball. The algebra of the first two of these is thoroughly covered in Harman, 1976, and Mulaik, 1972, while Rozeboom, 1991a, 1991b, amply details Hyball; so this needs only a bare-bones review here. But I must also say something about how the present study has implemented Orthomax/Promax and Oblimin computationally. For unless I can reassure you otherwise, you have every right to wonder if certain disappointing performances reported here might be due more to programming blunders than to method inferiority. Ultimately, you must simply take my word for it that I have done considerable cross-checking and detailed verification/correction of computations having suspicious-appearing output. But at least I can tell you the origins of my source code.

Note. When I describe the parameters that select variants of these rotation species, I shall not explicitly mention that an additional option common to all is Kaiser normalization, that is, temporarily rescaling each data variable to have unit common-part variance during rotation. The standard alternative to Kaiser normalization (NORM = 1) is leaving the items' variances at unity (NORM = 0) so that their common-parts variances remain communalities.

Orthomax is the family of orthogonal rotations whose most familiar variant is Varimax. Its measure of the quality of loadings on each factor f_j in a pattern matrix **A** is $Q_j = \overline{\mathbf{a}_j^4} - \gamma\cdot(\overline{\mathbf{a}_j^2})^2$, where $\overline{\mathbf{a}_j^r}$ $(r = 2,4)$ is the mean rth-powered loading in **A**'s jth column, and parameter γ is a non-negative scalar selecting variants of which Quartimax ($\gamma = 0$), Varimax ($\gamma = 1$), and Equamax ($\gamma = NF/2$) are most distinguished. Orthomax seeks to maximize $\overline{Q}$, that is, the mean of Q_j over $j = 1,\ldots,NF$. However, for uniformity with Oblimin and Hyball, whose quality measures are optimal at minimum, I shall define $\mathcal{L}_{\text{orth}}$ to be $1 - \overline{Q}$ and treat this as a Loss measure which Orthomax seeks to minimize. ($\overline{Q}$ never exceeds 1, and reaches this limit only when NORM = 1, $\gamma = 0$, and every variable loads on just one factor.)

Orthomax was executed in this study by code copied from IMSL with no significant modifications. This updated my older version thereof that proved to contain a convergence-check error invalidating my previous finding that Varimax seemed surprisingly sensitive to start position.

Promax (Hendrickson & White, 1964) has no pattern-quality measure of its own, but is parasitical upon some other method to provide a base rotation $\mathbf{A}_1$ of the input pattern which Promax then refines by procrustes rotation to a target pattern derived by raising each element of $\mathbf{A}_1$ to its Kth power ($K \geq 2$ a method parameter) while retaining its sign. Although any rotated pattern $\mathbf{A}_1$ can be Promaxed, its creators intended Promax to be a fast oblique refinement of an orthogonal rotation like Varimax preparatory to further oblique polishing by hand. So the present study tests Promax only with Orthomax base.

The Promax routine applied here is my own programming; but its logic uses only simple code techniques with which I have much experience. (That is, trust me.) Its version of procrustes is the computationally easy one that first rotates to least-squares match with target without concern for the rotated factors' variances, and only afterward normalizes the latter.

Direct Oblimin is a prominent family of oblique rotations that attempt, roughly speaking, to minimize a mean-shifted uncentered covariance between the magnitudes of loadings on the two factors in each factor plane.[1] More precisely, (direct) Oblimin's measure of pattern quality in the plane of factors $\langle f_j, f_k \rangle$ is $Q_{jk} = \overline{\mathbf{a}_j^2\mathbf{a}_k^2} - \gamma\cdot(\overline{\mathbf{a}_j^2})(\overline{\mathbf{a}_k^2})$, where $\overline{\mathbf{a}_j^2}$ is as above, $\overline{\mathbf{a}_j^2\mathbf{a}_k^2}$ is mean $\mathbf{a}_{ij}^2\mathbf{a}_{ik}^2$ over the items, and scalar parameter γ is in principle unbounded but in practice strongly recommended to be less than +.8 with nonpositive values preferred (Harman, 1976, p. 322). Oblimin's Loss function $\mathcal{L}_{\text{oblm}}$, which its computational routine seeks to minimize, is then mean Q_{jk} over all factor pairs $j \neq k$. Its most distinguished variant is (direct) Quartimin, picked by $\gamma = 0$.

Oblimin was executed in this study by modified IMSL code. In IMSL and other commercial rotation software, Oblimin iterates a series of conditional optimizations in which some factor f_j is shifted in just the plane of $\langle f_j, f_k \rangle$ for another factor f_k to the position that minimizes $\mathcal{L}_{\text{oblm}}$. (This planar shift primarily changes the loadings on f_k, but variance normalization also rescales the otherwise-unaffected loadings on f_j.) The IMSL code for this conditional (planar) optimization was copied with no alteration beyond minor housekeeping adjustments. But present code for managing iteration of these planar improvements expanded considerably upon its IMSL version: (1) Convergence of the rotation iteration was signalled by negligible change in factor position, the same as in Hyball, rather than by negligible decrease in the Oblimin Loss measure.[2] (2) Oblimin rotation from oblique start positions was enabled, as was also (3) rejection of rotations containing degenerately large pattern coefficients or factor correlations. And (4), provision was made for iterating the conditional optimizations in varied factor planes either serially, as standard for Oblimin, or in parallel as standard for Hyball. This very same global-management code also iterated the Hyball planar optimizations; so if any bugs survived its rather thorough vetting, they affected results from Hyball as much as from Oblimin.

Since Hyball rotation is my own creation, I can advise you with authority that its code in the present study was as good as it gets. But you still need a brief review of its logic. Hyball attempts to optimize the strength of hyperplanes by maximizing the density of near-zero pattern loadings while largely ignoring ones that appear to be salient. Its appraisal of a factor pattern **A**'s structural quality is grounded on a fulsomly parameterized hyperplane-misfit measure $\phi(e)$ whose arguments $e = |a_{ij}|$ are the sizes of individual loadings in **A**. The value of $\phi(e)$ starts at zero for hyperplane ideal $e=0$, and rises with increasing e to a finite asymptotic limit whereby as e grows large, $\phi(e)$ becomes increasingly indifferent to change in e. The pivotal parameter in ϕ is hyperplane-bandwidth *BH*; heuristically, item y_i is in the hyperplane of factor f_j just in case $|a_{ij}| \leq BH$. The curvature with which $\phi(e)$ rises as e increases from 0 to *BH* is determined by two parameters *JA* and *CV*, while a third, *JB*, selects the speed of approach to asymptote for loadings larger than *BH*. (See Rozeboom, 1991b, for details on these that you don't really need here.) Finally, Hyball's overall Loss function $\mathcal{L}_{hybl}$ is a weighted average of $\phi(|a_{ij}|)$ over all the loadings in **A**, wherein the differential severity of weighting is selected by a nonnegative parameter *WSAL*. Rozeboom, 1991b, describes the weights picked by *WSAL* > 0 as "salience" weights in view of their character in planar rotations; here, it suffices to say that the weight w_{ij} assigned to $\phi(|a_{ij}|)$ in $\mathcal{L}_{hybl}$ is the mean of $|a_{ik}|^{WSAL}$ over all $k \neq j$, this being simply 1 (no differential weighting) when *WSAL* = 0. When *CV* or *JA* is 0 with *WSAL* = 2.0 and *BH* very large (say 1.0 or more), $\mathcal{L}_{hybl}$ is almost though not quite identical to the Quartimin variant of $\mathcal{L}_{oblm}$. Hyball too attempts to **minimize** its $\mathcal{L}$ by iterating conditionally optimal planar shifts; however, when cycling through all factor planes it standardly concatenates its planar shifts in parallel rather than serially as clarified below. (Also unlike Oblimin, Hyball's immediate appraisal of a planar shift ignores any configuration changes that this shift entails in other factor planes. [For discussion of this point, see Rozeboom, 1991b, p. 194ff.] Also, Hyball planar rotations ignore points that could be brought into the hyperplane under adjustment only by an extremely large factor shift.) Finally, Hyball provides two modes of solution for its planar optima, SCAN (Brute-force scanning) and STEP (polished Step-down regression). STEP is much faster than SCAN, but not quite so accurate.

I have recently completed a massive appraisal of Hyball's method parameters' interactive success at source recovery from the same simulation datasets used in the present study. Although its quantitative findings are not now and probably never will be tidied for publication, their qualitative summary has been included in the documentation that accompanies my Hyball code package and highlights thereof should also be mentioned here. (1) SCAN is distinctly superior to STEP, but only modestly so. Their difference is largest at intermediate levels of hyperplane noise, and vanishes as hyperplanes become either very easy or very difficult to discern. (2) The lowest admissible value, -1, of curvature-within-hyperplane parameter *CV* is much inferior to $CV = 0$, which in turn is slightly inferior to $CV = 1$. There are hints that even larger *CV* may be trivially better yet; but differences over range $CV \geq 1$ should be so minuscule that overriding Hyball's default setting $CV = 1$ would seem pointless. (3) Results for curvature-within-hyperplane intensifier *JA* urge non-negative settings for this, but are ambiguous about its performance over range 0 - 6. On balance, however, the nod goes to *JA* setting 1 or 2, with 1 perhaps slightly the better under SCAN. (4) Variation in outlier-emasculation parameter *JB* over its three tested settings 2,4,6 affected performance only weakly. But in SCAN mode, *JB* setting 4 was distinctly superior to 6 and slightly inferior to 2; whereas under STEP, *JB* settings 4 and 6 were both modestly better than 2. (5) Not surprisingly, the best setting for hyperplane-bandwidth parameter *BH* is influenced--though less than you might expect--by the target hyperplanes' diffuseness. For the present ensemble of source patterns, *BH* in the range .15 - .25 is clearly preferable with

BH = .20 (or thereabouts) probably the best default. (6) Results for weighting parameter *WSAL* over tested settings 0, 1.0, 2.0 have surprised me. *WSAL* = 2.0 was substantially inferior to the others under all examined conditions; and in STEP mode, setting 0 was mildly better than 1.0. But in SCAN mode, *WSAL* = 1.0 outperformed *WSAL* = 0 consistently and appreciably. (*WSAL* = 1.0 under SCAN also appears slightly better than *WSAL* = .5.) So I now recommend *WSAL* = 0 under STEP but *WSAL* = 1.0 under SCAN.

Serial vs. Parallel iteration of planar rotation.

Orthomax, Oblimin, and Hyball all seek to locate their $\mathcal{L}$'s global minimum by iterating cycles of planar rotations over all factor pairs $\langle f_j, f_k \rangle$ wherein f_j is shifted to the position in the $\langle f_j, f_k \rangle$ plane that locally minimizes $\mathcal{L}$. In Serial iteration, the pattern/covariance change entailed by each planar shift is unconditionally executed before moving to the next plane in the iteration sequence. But how should factor pairs be ordered in that sequence? The standard $\langle j, k \rangle$ order within each cycle has been to step j in an outer loop from 1 to *NF* while k goes from 1 to *NF* in an inner loop for each j. However, when endeavoring my first primitive version of Hyball many years ago in considerable innocence of analytic factor rotation's established technology,[3] I felt uneasy about prospects for bias and misdirection in any routing scheme not responsive to the problem's loss gradients. One would think that ideally, at each step of the iteration the routine should ascertain for each $\langle j, k \rangle$ the $\mathcal{L}$-decrement that would result from optimal rotation of f_j in the $\langle f_j, f_k \rangle$ plane in order to execute the planar shift with the largest gain. Computationally, this always-best-gain routing would be prodigiously wasteful. But an effective approximation to it utilizing the gain potentials from all planes simultaneously is the following: First, given pattern $\mathbf{A}_r$ on the factors F_r reached after r rotation cycles, find for each $k \neq j$ the coefficient w_{jk} for the rotation of f_j in plane $\langle f_j, f_k \rangle$ that locally optimizes the loadings on f_k. And collect these coefficients in a raw rotation matrix $\mathbf{W}$ whose jkth element is w_{jk} if $j \neq k$ or 0 otherwise.[4] Next, for some damping fraction $\delta \leq 1$ (a control parameter), let the refined rotation matrix be $\mathbf{R} = \mathbf{D}(\mathbf{I} + \delta\mathbf{W})$ where $\mathbf{D}$ is the diagonal matrix of scaling multipliers that normalize the variances of rotated factors $F_{r+1} = \mathbf{R}F_r$. Then the pattern after r+1 cycles of parallel rotation is $\mathbf{A}_{r+1} = \mathbf{A}_r\mathbf{R}^{-1}$.[5] This procedure is similar to the steepest-descent method of nonlinear optimization, and insures that results would be unaffected by permutations of axes before or during the iteration.

Oblimin can be programmed just as easily as Hyball to maximize its $\mathcal{L}$ by Parallel rather than Serial iteration; and the present study has afforded a splendid opportunity to appraise the difference in yield between these two rotation styles over many varieties of oblique rotation.

<u>Spin Search</u>.

Hyball's Spin-search enhancement (Rozeboom, 1993) effectively obviates the influence of start position on rotation results while also identifying a plurality of axis positions that merit interpretive consideration. Although I take considerable pride in this routine, its logic is child's play (at least for children with multivariate precocity) and can easily be made an option in any modern rotation program. In the version of Spin search I now favor, the subroutine that controls this first executes a series of Tries, each of which randomly shifts ("spins") whatever axis positions happen to be current, rotates this Spin start to convergence under the program's current choice of method parameters, and stores the resultant factor pattern in a buffer file. (The occasional Try that stumbles into some degeneracy is re-started.) This continues until either MAXTRY Spin solutions have been collected, or the last NUFF Tries have failed to improve on the $\mathcal{L}$-wise best result obtained previously in this series. (In the present study, MAXTRY was set at 60 and NUFF at 30.) When Try collection has ceased, the stored Try patterns are ranked for quality under the current paramerization of $\mathcal{L}$, and the NSAV best of them are copied in order of their appraised quality to a relatively permanent log file. (In normal Spin search by Hyball, solutions sufficiently similar to better ones already logged are not saved. But in the present study, all ranked Try results were saved for further evaluation without similarity screening.)

<u>The Divergence measure of pattern similarity</u>.

The present study compares rotated factor solutions to one another and to the source structure by five measures of similarity applied after the comparison pair's factors are matched to maximize pattern agreement. These are (1) RMS (root mean square) difference between corresponding pattern elements; (2) maximum difference over the entire pattern between corresponding elements;[6] (3) mean (also minimum and maximum) congruence Divergence between the comparison pair's matched pattern columns; (4) RMS difference between the corresponding factor correlations (that is, the match on C_{FF}); and (5) maximum difference between the corresponding factor correlations. Except for (3), all these measures should be self-explanatory. But Divergence needs some clarification:

As a measure of similarity between two order-N pattern columns **x** and **y**, RMS(**x**-**y**) suffers interpretively from sensitivity to scale. Specifically, the RMS difference between c_x**x** and c_y**y** is much influenced by the signs and sizes of scaling multipliers c_x, c_y; whence **x** and **y** need some standardization of orientation and norm (euclidian length) before their RMS difference has clear significance. This standardization is nicely accomplished by the Congruence, $\text{Cng}(\mathbf{x},\mathbf{y}) = \mathbf{x}'\mathbf{y}/(\mathbf{x}'\mathbf{x}\mathbf{y}'\mathbf{y})^{\frac{1}{2}}$, between **x** and **y**, that is, the uncentered correlation between their corresponding elements. But Congruence, too, has a large interpretive defect in its insensitivity to mismatch at the high end of similarity. Thus the effect on Cng(**x**,**y**) of a given increment or decrement in RMS difference between standardized **x** and **y** is comparatively large when Cng(**x**,**y**) is small, but decreases to vanishing as the Cng(**x**,**y**) level on which this change is imposed approaches unity. This obfuscation can be nicely expunged, however, by nonlinearly rescaling Congruence as Divergence, namely $\text{Div}(\mathbf{x},\mathbf{y}) =_{\text{def}} \arccos(|\text{Cng}(\mathbf{x},\mathbf{y})|)$ with the angle measured in degrees for greatest familiarity. The Divergence between conforming vectors **x** and **y** is simply the acute angle between their axes in the spatial model of a vector configuration. Appendix A shows that when **x** and **y** have positive congruence and the same norm, their RMS difference is approximately $.017 \times \text{RMS}(\mathbf{x}) \times \text{Div}(\mathbf{x},\mathbf{y})$ with extremely high accuracy for Div less than 60°, and passably close even for Div up to 90°. Moreover, when **x** is a column of a source pattern or decent rotation estimate thereof, it is reasonable to expect RMS(**x**) to lie roughly between .2 and .4, whence $100 \times \text{RMS}(\mathbf{x}\text{-}\mathbf{y})$ should generally be from .3 to .7 times as large as Div(**x**,**y**), the lower ratio for weak factors and the higher one for heavyweights such as in the present study.

This idealized relation between RMS difference and congruence Divergence does not, however, entail that for comparison of factor patterns these differ only by a scale shift. For under standard variance normalization of items and factors, the norm of a recovered pattern column is affected not only by the corresponding source loadings but also by inaccuracy in the factor's recovered correlations with other factors. So Divergence is in principle a purer measure of pattern similarity than is RMS difference, and in my subjective impression of the results reported below exhibits their regularities more cleanly than does the latter albeit not enough to affect any of the conclusions that emerge.

RESULTS

The preliminary phase of this study examined source recovery by a spectrum of method variants in each rotation family from two standard start positions (no Spin search yet). Nine variants each of Orthomax, Oblimin, and Hyball were picked by a generic parameter $\delta\delta$ that stepped from 0 to 4.0 in intervals of .5 while selecting $\gamma = \delta\delta$ for Orthomax, $\gamma = -\delta\delta$ for Oblimin, and the assorted combinations of *BH* and *WSAL* itemized in Table 1. Each $\delta\delta$-selected Orthomax variant was also refined by Promax at each of power levels 2,4,6,8; while each $\delta\delta$-selected variant of Oblimin and Hyball was executed both by Serial (S) and by Parallel (P) iteration. Moreover, each of these procedure combinations was crossed with the two Kaiser-normalization alternatives and started both from the principal-factor extraction pattern and from the input's rotation by NORM-1 Equamax. (Why the latter rather than some other Orthomax start will be apparent shortly.) Each of these $2\times2\times9\times(1+4+2+2) = 324$ rotation variants was applied to each of the 100 extraction patterns recovered from the 100 simulation datasets, and their accuracy at source recovery tabulated separately for each hyperplane-noise level (20 patterns in each) as well as for all noise levels combined. The aggregate mass of these preamble statistics verges upon overwhelming; and since the main show is yet to come, I will spare you the Maximum-error appraisals and hyperplane-noise breakdowns beyond some fragments thereof in Table 1D that you can ignore for now.

Table 1 about here

Evaluation of these preliminary results in Table 1 can best begin with contrast dimensions that show the least interaction with others, and then focus on the latter at levels of the former that produce superior results.

<u>Start position</u>. One question decisively answered by Table 1 is whether start position matters. It does indeed, at least for Oblimin and especially Hyball. Although some of the differences for serial Oblimin are quite small, results are always better when started from the extraction axes' Equamax rotation than directly from the input position. So apart from the Orthomax variants, whose performance as continuations of the same Orthomax beginning is not what interests us here, we may as well focus on Table 1's Equamax-start sections when appraising the other contrasts.

NORM. Table 1 shows Kaiser normalization to be largely a triviality. For Orthomax, NORM=1 has a small advantage across all variants examined. But for Oblimin, the tabled NORM differences are negligible except for a tiny NORM-1 gain under serial iteration at the smallest γ settings. Hyball on the other hand prefers NORM=0 for benefits that are minuscule for all but its least successful variants. Table 1 cannot speak for all source structures; but to the extent that present results are generalizable, it makes no real difference how we go on Kaiser normalization.

Parallel vs. Serial iteration. For Oblimin from extraction start, parallel iteration is a disaster for all but the smallest γ (= $-\delta\delta$); whereas for Oblimin from Equamax start, parallel is persistently even if trivially better than serial at pattern recovery. (The preference order is reversed, still trivially, for covariances.) And for Hyball, too, the persistent superiority of parallel iteration is almost negligible. It appears, therefore, that my distrust of serial iteration has been baseless. Even so, modest but instructive differences between parallel and serial iteration will emerge from the main study.

Orthomax variants. Orthomax performance from extraction start is considerably influenced by choice of γ (= $\delta\delta$). (For this comparison, Equamax start is a contaminant.) Quartimax (γ = 0) was poorest by far; but Varimax (γ = 1.0), too, was appreciably inferior to the best results in a broad γ interval starting at or near Equamax setting γ = 2.5. In a side study, I have also tested γ-range 4.0 - 13.0 in steps of 1.0 and from 10.0 to 90.0 in steps of 10.0. The scarcely noticeable error decline as γ increases beyond 2.5 bottoms out in the vicinity of 4.0 and slowly--very slowly--increases as γ becomes large, reaching about the same accuracy at γ = 90 as Quartimax. So present results support past intimations (Harman, 1976, p. 299) that Equamax is the Orthomax of choice among name brands thereof, but suggest that $\gamma = NF/2$ may only be threshold to a band of mildly better γ settings. Be that as it may, Equamax's strong showing with the present data has motivated choosing this variant of Orthomax to provide Table 1's start alternative to extraction axes. (I have also run this analysis from Varimax start; and as you can see from the fragment of those results included in Table 1D, Varimax start was inferior to Equamax start, especially for Hyball.)

Promax variants. Table 1 indicates that although the accuracy of Promax pattern recovery diminishes as its powering parameter increases, the superiority of power 2 over power 4 is trivial.

But recovery of the source correlations is less tolerant: 4 is appreciably inferior to 2 in Table 1C, and higher powers continue the decline. These differences were confirmed by a side study that tested Equamax-based Promax under powers 2, 3, and 4 by Spin search. Powers 2 and 3 were indistinguishable and only trivially better than 4 at pattern recovery, but there was a small but noticeable loss of covariance accuracy with each power increment.

Oblimin variants. Under both NORM alternatives and from both start positions in Table 1, performance of parallel Oblimin deteriorates as its Gamma parameter increases in negative size from Quartimin setting $\gamma = 0$. The same is true of serial Oblimin except for hints that $\gamma = -.5$ and $\gamma = -1.0$ may be as good as or even better than Quartimin in this case. These hints are not specious; for Spin search has confirmed them. (Though essentially trivial, the improvement is reliable.)

> In a side study, I have also examined Equamax-started Oblimin under positive Gamma in steps of .1 from 0 to .8, steps of .5 from 1.0 to 4.0, and steps of 2.0 from 4.0 to 16.0. With increasing positive γ, performance deteriorated from Quartimin scarcely at all until γ passed .5, at which point degenerate rotations began to occur. Almost all were degenerate for γ between .6 and 3.0, but the likelihood of that reverted to negligible when γ reached 4.0. However, inaccuracy was then over twice that of Quartimin; and while success improved somewhat as γ grew large, it never achieved the accuracy of Quartimin albeit under NORM = 1 it came reasonably close.

Hyball variants. Compared to my unpublished study of Hyball's control parameters whose findings were briefly summarized above, the comparisons among Hyball variants in Table 1 are crumbs not worth careful mastication. But since the numbers are there, you may as well note the following: (1) All Hyball variants do substantially poorer from extraction start than from Equamax start, some startlingly so. (This is not typical of Hyball; it largely reflects the exceptional difficulty of these source patterns.) (2) From Equamax start, the Hyball variant most resembling Quartimin (picked by $\delta\delta = 4.0$) is distinctly the poorest tested here. But more provocative is that it is not quite so good as Quartimin. I have been unable to identify precisely what difference in the solution process gives Quartimin its edge. (3) Over Table 1's *BH*-crossed-with-*WSAL* Hyball variants, *WSAL* = 1.0 is consistently best of the tested salience weights and, excluding special case $\delta\delta = 4.0$, $BH = .15$ is worst of the three tested hyperplane-bandwidth settings.

[Orthoblique. I have also tested IMSL code for Orthoblique rotation as, in effect, an additional column of the Table 1 layout with Equamax for the orthonormal component of its rotation and generic variant index $\delta\delta$ calling its obliquity parameter C in steps of .10 from 0 to .80. From principal-factors start (which Orthoblique's theory presumes), the best setting of C was consistently around .4, yielding accuracies very close to Equamax-based Promax at power 2 or, for pattern recovery without Kaiser normalization, power 4. From Equamax start, Orthoblique becomes essentially an orthogonal rotation (C doesn't matter) with the same results as Equamax.]

Species comparisons. The Orthomax/Promax/Oblimin rotation variants appraised in Table 1, supplemented by my qualitative report on Orthoblique, chart performance on nearly all rotation options available in commercial multivariate software packages. If your choices are restricted to the latter, Table 1 says that you should prefer NORM-1 Equamax for orthogonal rotation, and for oblique solutions--here is a big surprise--continuation of Equamax by power-2 Promax. But Quartimin or perhaps even better $\gamma = -1$ Oblimin (under Serial iteration, since Parallel is not commercially available) is essentially as good as optimal Promax, especially if Quartimin/Oblimin is run from Equamax start with Kaiser normalization. Of course, it remains to be seen if these comparisons generalize reliably to recovery of source structures with configurations substantially different from the present simulation data. But Table 1 is a good provisional basis for choice among commercial rotation options.

On the other hand, if your yen for superior results is strong enough to motivate investing the bargain price ($10) and modest effort to install and learn my Hyball package for DOS or UNIX operation, you can appreciably improve upon the best Promax or Oblimin by use of Hyball from Equamax start. Moreover, Hyball will also allow you to invoke Spin search, the benefits of which will be examined next.

Optimization of Source-structure Recovery by Spin Search.

None of the performances recorded in Table 1 indicate source recoveries whose inaccuracies would not seriously jeopardize interpretation. Indeed, the threat of this is considerably worse than the Table 1 numbers make explicit. For a factor's interpretation is usually driven by its largest pattern loadings, and the maximum error in a Table 1 pattern estimate is on average about three times as large as its RMS error.

> This is roughly what you would expect if signed pattern errors are more or less Normally distributed around an expectation of zero, and was manifested quite consistently over all method variants and hyperplane-noise levels in the present study. In some later Spin-search runs, I have also collected the 95th within-pattern percentiles of error magnitudes, and find their average for each rotation variant at every W-level remarkably close to 2.0 times the pattern's RMS error, just as the Normal error model would predict.

Moreover, Rozeboom 199? reports evidence that for Hyball, at least, pattern-estimate error is essentially unbiased and dependent at most trivially on loading magnitude. So when an estimated pattern similar in size to the present 25×5 has an RMS error exceeding .15 or so, there is a good chance that some of its most conspicuous loadings are imposters while a few that by rights should be interpretively pivotal have dropped from sight. Is this the best that we can do when source structures become complex? With any luck, the future will continue to provide advances; but one prospect for improvement is already at hand: Spin search. To the extent that a rotation variant appraised in Table 1 is sensitive to start position, its performance should be correspondingly enhanced by any procedure that can locate and report its best local optima. But is Spin search really worth its considerable increase in computation time? Consider the following results and decide for yourself.

The findings now to be reported were obtained by repeated runs of a program that for present purposes I will call SPINTEST. Each SPINTEST run collected information on the performances of (generally) six rotation variants, one $\delta\delta$-selection each of Orthomax, Promax, Oblimin, and Hyball with both Parallel and Serial iteration of the last two. Under each selected Variant, SPINTEST rotated each of the 100 input patterns by a series of Spin Tries--random axis shifting followed by rotation to criterion and temporary storage of the result--terminated either after 60 Tries or when the last 30 Tries failed to improve on the preceding best in this series. These collected Tries were then ranked for quality on this Variant's Loss measure, Rank 1 being the series' solution identified as optimal by this criterion.

The Spin shifts commencing Tries were orthogonal for Orthomax, but generally oblique for Oblimin and Hyball. Each ranked series for Promax was obtained by applying Promax to each Spin solution in the corresponding Orthomax ranked series.

In addition, the ranked collection from each Spin series was augmented by this Variant's "Rank 0" solution obtained by ordinary rotation (no spin) after shifting the input axes to procrustes alignment with the source pattern or, for Orthomax/Promax, after Equamax pre-rotation. (For Oblimin and Hyball, this Rank-0 solution should closely approximate the best solution this Variant can find by Spin search; for Orthomax/Promax, it is only a surrogate near-best that discussion will ignore.) Finally, the ranked Tries in each Variant's Spin collection for each input pattern were appraised for similarity to one another and to the source structure, with accumulation of these appraisals in summary tables. Comparisons to source used all five similarity ratings cited in the introduction, namely, RMS and largest difference between matched pattern loadings, congruence Divergence between matched pattern columns averaged over factors, and RMS and largest difference between matched factor correlations. For brevity, I shall henceforth refer to these five appraisals of a solution's success at source recovery as *Sim* measures. Consistency comparisons of solutions within Spin collections examined only divergence; but in addition to the mean column divergence between compared Spin patterns, the minimum and maximum of their matched-column divergences were also tabulated.

More specifically, the output of each SPINTEST run extracted the following information from the results in each ranked Spin series. For each tested method Variant, means and standard deviations on these performance appraisals were compiled separately at each hyperplane noise level (20 input patterns each) as well as over all combined.

1. Start-position sensitivity (Table 2). Present adjudication of this matter has sought to determine (*a*) how similar on the whole are a Variant's rotations of the same input from independently random starts; and (*b*) how similar among many Spin Tries of this Variant are the ones to which the Variant's criterion gives highest preference. Although quantitative answers to these questions are of necessity strongly conditional on the patterning latent in the particular factor solution being rotated, much can still be learned in this respect from the present ensemble of source structures. Question (*a*) is answered here by the minimum, mean, and maximum ("Min/Av/Max") column Divergence

between a Spin series' first and last unranked Try patterns. (Any other two Try selections unselected for rank should do as well.) And question (*b*) was addressed by determining, for each of the ten $\mathcal{L}$-wise best solutions in each ranked Spin collection, its Min/Av/Max Divergence from the collection's lower-ranked solutions, including Rank 0. However, since no trends nor much difference in these measures over ranks 1-10 was apparent to eye, they have been condensed in Table 2 into the mean Min/Av/Max Divergence among all the Spin series' solutions with ranks no greater than 10.

2. Accuracy of source recovery (Table 3). From each ranked Spin collection (one for each Variant with each extraction pattern) seven not-necessarily-distinct patterns were selected for special attention: (1) its *deus-ex-machina* Rank-0 estimate of the Variant's global optimum for this input; (2) its Rank 1 genuine Spin solution, that is, the one judged optimal by the Variant's operational criterion; and (3-7) for each of the five *Sim* measures, the Spin solution in this collection that was in fact Best in its so-appraised match to source. Unlike Rank-0 and Rank-1 solutions, a *Sim*-wise Best solution's rank in its Spin collection is an additional recovery datum that proves to be of considerable interest. For each *Sim*, SPINTEST compiled a subtable averaging over the 20 ranked Spin collections at each *W*-level for each method Variant the *Sim*-wise accuracy of the Rank-0, Rank-1, and *Sim*-wise Best solutions, as well as the rank-in-collection of this Best.

3. Special comparisons (Table 4). Since a Spin collection's solution identified as optimal by a method Variant's $\mathcal{L}$-measure is not always its Try result that in fact most closely matches the source structure in one or another distinguished respect, it is of interest to observe how closely the first resembles the others. So SPINTEST has also extracted from each Spin collection the Min/Av/Max pattern Divergences and RMS covariance differences among its Rank-1 solution and its Try results that respectively differed least from source in RMS pattern error, mean pattern Divergence, and RMS factor-covariance error. Low disagreement ratings here can result either from the same solution being picked as best in these assorted respects or from close resemblance of different picks. Since much of the information obtained on these special comparisons has rather limited value, only its more interesting portions are reported below. (Actually, what Table 4 reports is a small modification of the special comparisons just described in light of findings on the original version. Details later.)

Although I have run SPINTEST under many choices of its control options, the variant of each rotation species on which I have focused, the only ones reported here in detail, are: Orthomax with $\gamma = 2.5$ (Equamax); Promax at power 2 from Equamax base; Oblimin with $\gamma = 0$ (Quartimin); Hyball in SCAN mode with <JA, JB, BH, CV, WSAL> = <1, 2, .20, 1.0, 1.0>; and NORM = 1 for all, since Orthomax and Oblimin show a slight preference for Kaiser normalization in Table 1 while Hyball doesn't care. Even though each cell in SPINTEST's printout gives the mean (and with a few exceptions the standard deviation) for a particular aspect of some method's response to the 20 input patterns at a given noise level, or 100 for all *W*-levels combined, almost all these means are products of Spin search and perforce contain some chance departure from their statistical expectations on this database. So when comparing one SPINTEST mean to another, one would like some indication of their sampling error. Accordingly, the means reported in Tables 2,3,4 are actually averages over the corresponding means in 10 repetitions of SPINTEST on the rotation Variants indicated, followed in parentheses (except for Table 2) by 10 times the standard deviation of those 10 means. Thus for any cell entry of form "m (s)" here, the estimated sampling error of mean m is $(s/10)(10-1)^{-\frac{1}{2}} = s/30$, whence $m \pm s/10$ is an interval estimate for m's expectation at confidence level 99%.

Once the meanings of their entries become clear, Tables 2-4 pretty well speak for themselves. Even so, I had better walk you through their highlights. Let us start with Table 2's message on the sensitivity of rotation results to start position, looking first at its "Unord spin" columns which tell how much Divergence to expect between two arbitrarily selected Spin rotations by a given method of an input pattern having the present study's source structure at the indicated level of hyperplane noise.

Table 2 about here

Method differences in start sensitivity.

There are some very large contrasts among Table 2's Unord-spin ratings, the largest of which distinguish Equamax/Promax from Quartimin and Hyball. Equamax (and as a consequence Promax) is not merely insensitive to start position in the present inputs, it is astonishingly so. Pattern differences no greater than the largest tabled entry for Equamax (on average, only 1.5° divergence between most poorly matched factors at *W*-level .20) should be visible only in the patterns' 3rd decimals; and most Equamax Spin rotations from the same input are much closer than that. I have also confirmed that Varimax too is this consistent under Spin search, so we can expect the same of all variants of Orthomax and other methods (notably Promax and Orthoblique) grounded on an Orthomax criterion.

This high-grade identity of Equamax/Varimax output no matter where started seems almost too ideal for belief. My own past observations of startlingly large Varimax sensitivity to start position can be written off to the now-corrected code error confessed earlier. But that does not account for Cattell & Gorsuch's original (1963) discovery of such indeterminacy under Varimax. And in a cheerful letter elicited by my Spin paper, Mark Foster (Research and Evaluation, Colorado Division of Mental Health) has advised me that he and a colleague reported similar findings in a paper presented to the 1971 Rocky Mountain Psychological Association meeting (didn't anyone care??). He also alerted me to a paper by Gebhardt, 1968, who contrived a 12×4 factor pattern yielding two distinct Varimax solutions. When I tested NORM-1 Varimax Spin search on the Gebhardt pattern I found not just two but six local optima, all but one with mean divergence over 13° from one another and most over 20°, that randomized starts can repeatedly recover. Yet extended Spin search of this same pattern by NORM-1 Equamax homed in on Gebhardt's original with 100% consistency. I have also re-examined (after bug extermination) Orthomax performance under Spin search on the classic nightmare of Thurstone's 26-variable Box problem (see Rozeboom, 1992, p. 587ff.) and find that in this case it is Varimax that always yields the same solution (though not a particularly good one) no matter where started, whereas Equamax finds two that diverge widely on all three factors not only from each other but also from the Varimax solution. Even so, one can evidently rotate under Orthomax with high even if not altogether certain confidence that the solution returned is globally optimal on $\mathcal{L}_{\mathrm{orth}}$.

Whether we should *want* our rotation algorithms to be this Spin-invariant is an issue still needing adjudication. But asset or drawback, Table 2 makes plain that it is not shared by Oblimin or Hyball. Serial Quartimin's start sensitivity with the present data is fairly mild. But it is still large enough to range over solutions urging appreciably different interpretations of some factors. And results for parallel Quartimin as well as Hyball in both iteration styles are strongly affected by start position. Here are some more specific contrasts worth noting in Table 2's Unord-spin report for Quartimin and Hyball: (1) Comparisons over the Min/Av/Max divergence measures reveal enormous within-pattern variation in start sensitivity: Some factors (or more precisely the loading columns that demark them) within a pattern recovered by an optimization routine are generally *much* less Spin-invariant than others. Presumably this primarily reflects chance departures from expectation (the same for all factors) under the frame parameters that constrained each dataset's random production. But it demonstrates how delicate are the conditions of decent factor recoverability. (2) Although start sensitivity increases with hyperplane noise, the proportionate increase with each W-step is relatively modest. (3) Parallel iteration yields greater start sensitivity than does serial iteration. Why this difference is so much greater for Oblimin than for Hyball, I cannot explain. But its existence is part of a story that will unfold as we continue.

The pattern of contrasts in Table 2's Ranks 0-10 columns is largely the same as in its Unord-spin columns, but there are significant differences in detail. (1) When only a small number of $\mathcal{L}$-wise best rotations are retained from an extended Spin series, the chosen few--call these the Spin series' *Cream*--are considerably less divergent than are two solutions picked at random from the series. But the Spin Cream of a method whose global optimum is unique and attainable from many diversified start positions should contain scarcely any differences at all; so what is most striking about this part of Table 2 is how much divergence persists even in the Cream of Quartimin and especially of Hyball. (2) The effect of hyperplane noise on Cream divergence is quite small for parallel Oblimin and almost negligible for serial Quartimin. In sharp contrast, Cream solutions from both parallel and serial Hyball are nearly identical at the three lowest W-levels apart from some mild disagreement on the worst-matched factors at W=.10; but as hyperplane noise increases beyond that, their similarity deteriorates explosively on all but the best matched factors, which remain highly consistent even at W=.20.

Table 3 and Figure 1 about here

Table 2 makes clear that for Oblimin and Hyball, start position can strongly affect the factor positioning these yield. Table 3 and its comparison to Table 1D, partly depicted in Figure 1, show what can be gained in accuracy of source recovery when the method's top Spin Cream (Rank 1) is chosen for interpretation. Several important conclusions emerge from Table 3, of which the most beguiling is the least apparent and will be taken up last.

Method differences in Spin achievement.

Start with Table 3's Rank-1 columns, which show for various combinations of rotation method, hyperplane noise, and *Sim* measure how well this study's $\mathcal{L}$-wise best Spin solutions in fact recovered the source structures. As a preliminary, note that Rank-1 Spin Cream from Hyball is consistently even if minutely more accurate under parallel iteration than under serial, whereas under Quartimin the reverse is true. That is, the operationally superior iteration style in each case is the one that is standard in this method's distributed software. So hereafter, when I speak of comparisons involving Hyball or Quartimin without explicit mention of iteration style, parallel Hyball and serial Quartimin are to be understood.

The first salient point about method-conditional Rank-1 accuracy, detailed in Table 3 with highlights in Figure 1, is the effect of hyperplane noise. For all methods on all similarity measures, error increases monotonically (apart from a few minor inversions under Quartimin) with W-level. For Equamax/Promax and Quartimin, the rate of increase is rather small. But it starts at $W=0$ for them with recovery errors that are already too large for reliable interpretation of results. In marked contrast, Hyball starts at $W=0$ with accuracy nearly at its procrustes-estimated theoretical limit, and deteriorates with increasing W at a pace which at first is almost negligible but steepens sharply beyond $W=.10$ until at $W=.20$ Hyball does no better--in fact, on some of the *Sim* measures considerably worse--than Promax and Quartimin. This is more or less what could have been predicted, since $\mathcal{L}_{\text{orth}}$ and $\mathcal{L}_{\text{oblm}}$ stress the largest pattern loadings while largely ignoring the small ones, and just the opposite is true of $\mathcal{L}_{\text{hybl}}$ under its recommended parameter variants. But so forceful a demonstration of payoff from this shift in criterion logic is edifying. And its implications for applied factor rotation are plain: For source structures of considerable complexity, neither Orthomax/Promax nor Quartimin can be expected to find a positioning of axes in extraction space on which item weights match the source pattern with better than crude accuracy no matter how sharp the source hyperplanes may be. But source complexity is no impediment to near-perfect pattern recovery by Hyball so long as hyperplanes are sufficiently clean. Nevertheless, Hyball cannot discern the indiscernible; so as hyperplanes become increasingly indistinct, Hyball's superiority over other rotation methods dwindles to nil. When choosing axes for a factor space seemingly deficient in quality hyperplanes, present results suggest that Equamax-based Promax may be our best choice of rotation method. But in that case we can't expect to learn much about the source structure no matter how we rotate.

A second foreground issue illuminated by Table 3's Rank 1 findings with an assist from Table 1D is what top Spin Cream gains in recovery accuracy over simple rotation from Equamax start. Several surprises lie in this. Most unexpected for me was the modesty of Spin's superiority for Hyball, since as reported in Rozeboom, 1993, my incentive for developing Spin search was difficulty in achieving decent standard-start source recovery by Hyball from data with essentially the same structure as the present $W=0$ datasets. (The error in my Orthomax code at the time had much to with that ineptitude.) Indeed, one might wonder if the performance gain shown by the gap between Hyball's solid and dashed lines in Figure 1 is large enough to warrant the considerable time cost of Spin search. The answer: Defiantly YES, at least in the final stages of choosing axes for interpretation and factor-score estimation. Spin's enhancement of Hyball's accuracy at the intermediate W-levels here, albeit small, is far from trivial. And more strategically, although Equamax proves to give an excellent start position for the present source structures, there are many other variants of Orthomax not to mention other standardly available rotation methods that can also provide start positions probably surpassing extraction start in many applications and perhaps improving at times upon Equamax as well. Why dither over which one to use and risk a poor choice when Spin search obviates the start problem?

> As mentioned earlier and documented in Table 1D, Varimax starts were appreciably inferior to Equamax starts here, just as you would expect from the extraction-start Orthomax results in Table 1. But extraction-start performance in Table 1 is not a fully reliable guide to best choice of single-try start position for Hyball: I have also tested Equamax-based Promax(2) for Hyball start, and find it to be slightly inferior for this to plain Equamax.

Another large surprise in Table-3/Figure-1 is that Quartimin's Rank-1 Spin Cream has considerably *less* source-recovery accuracy than its rotations from Equamax start. Strange as this may at first seem, it makes good sense when one reflects that for any $\mathcal{L}$ a rotation method may elect to optimize, the rankings of alternative rotations on $\mathcal{L}$ will correlate at best imperfectly with their rankings on match to source.[7] Thus some axis positionings may well be *Sim*-wise superior to the one that globally optimizes $\mathcal{L}$; and if the solution iteration is started near one of the former, it might be trapped by a local optimum before its accuracy sinks to that of the latter. This appears to be what has happened with Quartimin in the present applications: The axes at $\mathcal{L}_{oblm}$'s global optimum do not match the source structure as well as do axes at some of $\mathcal{L}_{oblm}$'s local optima; and the Equamax solution is generally in the capture region of one of the latter.

An aside on Rank 0 results.

This imperfect correspondence between best on $\mathcal{L}$ and best on *Sim* has other manifestations in Table 3. One is in the Rank-0 results which, you may recall, are *Sim* ratings of rotations started at the procrustes approximation to the source pattern. Earlier, I introduced these with the remark that they should closely approximate the best solution that Spin search can achieve for the data and method variant tested. This left ambiguous whether $\mathcal{L}$-wise best or *Sim*-wise best was envisioned, but neither reading proves to be altogether correct. Although Table 3's Rank-0 means for Quartimin and Hyball are almost everywhere superior to their Rank-1 counterparts, some dramatically so, these Rank-0 results are *not* asymptotes to which Rank-1 accuracies would converge with sufficiently extensive Spin search. A test of this possibility with extremely prolonged Spin search (NUFF = 300, MAXTRY = 1000) achieved no gain in Rank-1 accuracy though it did slightly improve the Bests. But more informatively, SPINTEST also reports for each of its Spin collections the number of Tries therein having $\mathcal{L}$ ratings lower than its Rank-0 solution. And the means of these counts over all datasets for Qmin-P, Qmin-S, Hybl-P, and Hybl-S were respectively 16.1, 22.5, 6.2, and 5.7. So regardless of whether Spin search hits upon the Rank-0 solution, it generally finds several less accurate solutions that $\mathcal{L}$ likes even better. Neither are rotations from procrustes start generally the most accurate that Spin search can achieve: Comparisons of the Rank-0 and Best columns in Table 3 show Quartimin and Hyball usually attaining better *Sim* ratings by Spin search than by rotation from procrustes start. Only on the pattern-match *Sims* under parallel iteration is this superiority order prevailingly reversed--trivially for Hyball, rather less so for Quartimin. (No large insights are evident in these comparisons, but patterns of results deserve notice even when of dubious importance.)

The challenge of *Sim*-wise best Spin Tries.

If optimizing a rotation criterion $\mathcal{I}$ does not generally find the very best approximation to source available in the extracted factor space, we can scarcely expect a routine for optimizing $\mathcal{I}$ to find solutions superior to that as well. Yet Spin search makes that possible; and the three rightmost columns in Table 3 attest that this possibility is eminently realizable in practice. The secret of this legerdemain lies in accompanying report of the routine's solution for the global optimum with a ranked selection of the merely-local optima it has identified as well. For some of the latter may well be appreciably better solutions than the one favored by $\mathcal{I}$.

There are, of course some formidable obstacles to utilizing these local optima effectively. But before probing prospects in that regard, we should ask whether anything in them appears worth the effort. Accordingly, consider the difference between the Rank 1 and Best columns in Table 3 for Hyball and Quartimin. In all cases (or at least on average within *W*-level), the Spin rotations most resembling source on the *Sim* at issue surpassed the $\mathcal{I}$-wise favored rotation in this respect. For Quartimin, Best is substantially better than Rank 1 at all *W*-levels, even better than its Equamax starts (cf. Table 1D). But that is only of incidental interest, since Quartimin is so far off pace in the accuracy chase. Hyball is the method that matters here; and in Table 3 and Figure 1 it is plain that Hyball's Best improves impressively on Rank 1 when hyperplane noise becomes troublesome. Admittedly, the solution that is best on one *Sim* is not generally also best on the others. (More on that below.) But wouldn't you be delighted to rotate under expectation of source recovery tracked by even one of the dotted lines in Figure 1?

To make actual use of such Best rotations, however, we must first pick them out of their Spin collections--which for empirical applications is a very nice problem indeed, insomuch as their $\mathcal{I}$ ratings do not suffice. I can think of one tactical and one strategic response to this challenge, neither very promising for routine practice but both worth research attention. Tactically in empirical applications, we can simply print out all patterns in our Spin Cream and spend considerable time studying them for interpretability and graphic appeal. This has two evident drawbacks, however. One is the formidable amount of work required to study the Cream in much depth. Happily, Table 3 indicates that for Hyball (unlike Quartimin), Best tends to occur decently high in the $\mathcal{I}$-wise preference order for the present datasets even if its rank is quite variable. Even better, Hyball's standard operation allows the user to filter out Spin Tries that diverge only trivially from lower-$\mathcal{L}$ solutions in the Spin series; and Table 3's rightmost column shows that when the present Spin collections are filtered to include only solutions whose Max Diverg from each lower-ranked solution in the filtered series is at least 5°, the mean and variability of the Best solution's filtered rank for the most part become comfortably small. And although this reduction is less than might be desired when hyperplane noise is severe, SPINTEST also reports (in output not tabled here) that approximately a third of the Spin solutions between Rank 1 and Best at the two highest *W*-levels are also *Sim*-wise better than Rank 1. So for applications within generalization range of the present study, subjective evaluation of Spin Cream in modest depth is likely to include the Best or at least some of the Better. Unhappily, the second drawback to this tactic is our having little reason for confidence that even experienced practitioners can discriminate Best from Pretty Good. Even so, when the factored items are drawn from a domain supporting a credible theory, some solutions in the Spin Cream may make rather more sense than others. In that case, one can favor Most Sensible over Lowest $\mathcal{L}$ with reassurance that since Best is probably not the latter it may well be the former.

Strategically, the disparity between Rank 1 and Best Spin solutions challenges us to seek further improvement in our pattern-quality measures. Given that some version of Thurstonian simple structure is for better or worse our best clue to the causal grain of common-factor space (perhaps not everyone will agree), present results make clear that large advances in analytic implementation of this notion beyond its first wave of development at mid-Century are not just possible but have already been realized in $\mathcal{L}_{hybl}$. But surely your faith that current $\mathcal{L}_{hybl}$ is the best we can do lacks conviction. My own efforts to find supplementary measures of pattern quality whose conjoining with $\mathcal{L}_{hybl}$ can further enhance source recovery have so far been a complete failure. (Additional measures, yes; helpful conjoinings, no.) I have not abandoned this search, but am much in need of some fresh ideas on what pattern features may be symptomatic of ideal axis positioning. I welcome your suggestions.

Table 4 about here

Some lesser findings on Spin-search.

More on profits and practicalities of replacing Rank 1 by Best. Even if there is no really effective way in practice to move from lowest-$\mathcal{L}$ to one or another Spin Best, one might still wonder how much difference it would make for what we get were we able to replace the former by the latter. To illuminate this, Table 4 shows W-stratified means on Min/Av/Max Divergence and RMS covariance mismatch among Source, Rank-1, DivP (Best on divergence from source pattern), and RmsP (Best on RMS difference from source covariances).

> Table 4's original design, intended to reveal the similarities among Rank 1 and the three most salient Best picks, had RmsP (Best on RMS pattern difference from source) in the slot now occupied by Source. But RmsP proved to match DivP so closely that clearly a Spin series' solution that was Best on one of these two *Sims* was almost always Best on the other as well. This was useful information; but once established, it left Table 4 massively redundant. (It also explained why the rightmost two columns of Table 3C are nearly indistinguishable from those of Table 3D.) Meanwhile, it became apparent that some breakdown of match-to-source finer than in Table 3 was desirable. Hence Table 4's present layout.

I shall limit my comments on Table 4 to Hyball-P. (Hyball-S is nearly the same, and Quartimin doesn't really matter.) As preface, note that the operationally identifiable Rank-1 and the theoretically preferable Bests are all (for Hyball) almost always identical at the two lowest W levels and still differ but little at $W = .10$. So only the larger-W rows of Table 4 have much to tell. These show first of all that if we could replace Rank 1 with a Best, choosing DivP would not only substantially improve match with source pattern but would also gain over half the improvement in covariance recovery afforded by RmsC; whereas picking RmsC would gain scarcely anything on pattern match. So replacing Rnk1 by DivP, which may at times be feasible, is much preferable to replacing Rnk1 by RmsC even if we had some way to accomplish that.

Secondly, Table 4 agrees with Table 3 (as it should, since portions of the former are included in the latter) that the pattern match of Rank 1 to Best for Hyball is superb at the three lowest W-levels, with just a hint of hyperplane blur beginning to trouble the worst-matched factors at W-level .10, whereas this near-ideal agreement deteriorates sharply as *W* increases beyond that. But Table 4 also apprises us that the large-*W* severity of mismatch among source, Rank 1, and the two Bests is not distributed at all uniformly over these solutions' pattern columns but is concentrated heavily in just one or perhaps two pattern columns.[8] So Hyball's performance at the highest two *W*-levels is not really as bad as Figure 1 makes it appear: If we could just ignore the one or perhaps two columns that are grotesquely in error, what remains of the solution would be quite decent; and neither would patternwise Best improve much on Rank 1.

Since this nonhomogeneity of pattern-recovery error may well owe much to the randomization-within-frame construction of the present source patterns, its generalizability to real data is especially soft. Even so, it demonstrates just how diverse the interpretive quality of a rotated pattern's columns can be.[9] But how can we best distinguish good pattern columns from bad ones in practice? Ultimately, this must be for experienced interpretation to decide. Yet Table 2 suggests a possible computational assist. We saw there that some factors in the Spin Cream are much more start-sensitive than others; and for each factor in each retained Spin solution, it is simple to record how tightly it agrees with its best matches in the others. To test whether a particular factor's Spin-invariance is usefully diagnostic of its pattern-recovery accuracy, I have persuaded SPINTEST to compute for each method variant at each *W*-level the linear correlation of a Rank-1 factor's divergence from closest source factor with its mean divergence from its best matches in the Spin series over Ranks 2-10. For Hybl-P, these correlations at *W* from 0 to .20 were .06, .16, .39, .39, and .35, respectively. The near-zero values at the two lowest *W* are neither surprising nor disappointing, considering Hyball's uniformly high accuracy there. But at the higher noise levels, where this aspirant predictor's help would be much appreciated, the correlations are still too low to have much practical value even if we could trust them to generalize. The admonition to take from this is that although the relative Spin invariance of particular factors is probably not in general entirely unrelated to their interpretive quality, it deserves only low weight when judging which pattern columns warrant the most respect.

<u>*Sim* similarities</u>. It will probably have occurred to you that the assorted *Sim* measures used in this study have seemed largely equivalent in what they tell, raising the question whether they are

essentially interchangeable apart from scale. To appraise this, SPINTEST has also determined the *Sim* correlations for each method variant (*a*) over all its Spin solutions at all *W*-levels combined and (*b*) over just the first 10 in each of its ranked Spin series after filtering. Table 5 shows these computed from raw moments accumulated over 10 SPINTEST runs. The correlations are mainly high, especially RmsP with DivP and RmsC with MaxC; but not so high that any is fully replaceable by another. In particular, the correlations between inaccuracy in recovering source-pattern and source-covariances are good but not great for Hyball, and rather mediocre for Quartimin.

Table 5 about here

The relation between pattern *Sims* and covariance *Sims* has considerable factor-analytic importance if identifying factors by pattern is hoped to reveal their location as well. For high accuracy of the item loadings recovered for a target factor is mathematically compatible with very large errors in that factor's recovered position in common-factor space as canonically defined, say, by its covariances with the factored items' common parts. That is, the quality of some column in a factor pattern is no guarantee of similar quality in the corresponding factor-structure column nor in the factors' estimated covariances with other distinguished dimensions of factor space. In particular, we can seldom if ever maximize match-to-source of factor pattern and factor covariances simultaneously. Yet doing better on the one should at least *tend* to improve on the other as well. Table 5 provides useful information on the strength of that tendency, but only at the resolution of whole solutions. To explore the within-solution relation of recovery accuracy on pattern vs. covariances for individual factors, SPINTEST has also determined, for each method variant at each *W*-level, the correlation over all pairs of different factors in the Rank-1 Spin solutions between the size of error in those factors' covariance and their mean divergence from the source factors they respectively match. For Hyball-P, these correlations at the assorted *W*-levels in ascending order were .10, .14, .45, .37, .24. (Hyball-P was roughly the same, while both Quartimins and Promax started in the .30s at $W=0$ and fell off appreciably after that. Although these correlations pooled raw moments over 10 SPINTEST repetitions, they still contain considerable sampling error which I have not tried to identify precisely but estimate to be in the vicinity of .05.) These numbers point to a worrisome situation whose discussion must be deferred to some other occasion.

The case for Parallel iteration.

So far, I have said little about output differences between Parallel and Serial iteration (henceforth PI and SI, respectively), suggesting by silence that there is little to choose between them. That is not so. PI proves to have an advantage over SI that in principal is rather important even though its gain in the present applications has been rather small. The effect is most conspicuous for Qmin-P vs. Qmin-S, but it is also reliable for Hybl-P vs. Hybl-S albeit considerably more subdued there: Parallel iteration enables Spin search to return a broader diversity of local optima than does Serial iteration, with the consequence that PI Cream is likely to enjoy lower $\mathcal{L}$-ratings than SI Cream, and hence, insofar as $\mathcal{L}$ is usefully diagnostic of the factor positions we hope rotation will attain, should yield the interpretively better results.

That Parallel iteration has the greater start-sensitivity is loud and clear in Table 2. All but one of the PI entries therein are larger than their SI counterparts, with the differences in Unord Spin persisting unabated if not intensified in the Spin Cream. For Hyball, these contrasts are often quite small; for Quartimin, they are enormous. I have not extracted the resultant quantitative $\mathcal{L}$-rating differences (apart from sign, the numbers wouldn't mean much); but their payoff is plain in Table 3. Look first at its report for Quartimin. All solution-quality columns except Rank 1 show greater accuracy for Qmin-P than for Qmin-S. And PI Bests better than SI Bests is just what should be expected from PI's greater Spin diversity. Moreover, Quartimin's prima facie paradoxical reversal of PI/SI superiority on Rank-1 accuracy can also be attributed to the broader scope of Spin returns under Parallel iteration. For if $\mathcal{L}_{qmin}$ correspondence with match-to-source is so poor that its local optimum closest on the iteration trajectory to Equamax start is generally *Sim*-wise better than local optima with lower $\mathcal{L}_{qmin}$ elsewhere, as argued earlier, we should not be surprised if Rank 1 accuracy deteriorates even more as Spin search pushes closer to $\mathcal{L}_{qmin}$'s global limit.

With a few trivial reversals, Table 3 shows this same pattern of PI/SI differences for Hyball, except that the contrasts are much smaller while as befits $\mathcal{L}_{hybl}$'s superior diagnostic accuracy, Rank 1 too is more accurate under Parallel than under Serial.

Some mystery still persists in Table 3's PI/SI contrasts, however. Their startlingly large difference in Quartimin incidence of low-ranked Spin solutions that fail to pass filter is simply another manifestation of Serial iteration's narrower Spin reach. But the argument from Spin scope doesn't explain the large superiority of Qmin-P over Qmin-S on Rank 0, nor why, unlike Qmin-S, Qmin-P's Rank 0 is always better than Spin Best on all the pattern *Sims* (but never on the covariance ones). Neither is it plain why the unfiltered rank-in-series of Quartimin's patternwise Best Spin solutions should be so much lower under PI than under SI. Possibly these are due to particularities of how $\mathcal{L}_{qmin}$'s local minima are scattered in the present datasets. But alternatively, might they not manifest intricacies of the solution mechanism which, were we to understand them, could enhance our proficiency in design of nonlinear optimization routines? Probably not; but I'm only guessing.

SUMMARY

A great deal of information on the performance of extant factor-rotation routines have been reported above, findings which if at all generalizable should have considerable value for guiding method preferences in practice. But all present solution-quality measures are aspects of match to one particular choice of axes in the dataset's common-factor space, namely, the factors randomly generated within the frame constraints detailed at this paper's outset. And had some other recovery target been stipulated, e.g. the population axes that minimize $\mathcal{L}_{qmin}$, Table 3 and Figure 1 would have looked very different. So why should these results be expected to generalize when they don't apply to other targets in the present datasets, and data in empirical applications will seldom have causal origins structured like the present production process?

Although the factors persistently referred to as "sources" here are, arguably, genuine causes of the factored variables in this study, that is not the main reason for choosing them as recovery targets on which to generalize. For those of us who believe that science advances far more successfully through explanatory induction from distinctive data patterning than by hypothetico-deductive tests of speculation (cf. Rozeboom, 1972, 1990), the features of an item configuration that most forcefully dictate where to position axes in their common-factor space are strongly demarked hyperplanes. This is not an abstract preference for an abundance of near-zero pattern loadings urged by faith that nature is frugal in causal connection, nor reluctance to take issue with Thurstonian orthodoxy on simple structure. Rather, like other basic explanatory inductions, it is a perceptual response to distinctiveness nearly at the level of animal instinct: When we study spatial models of a factor pattern containing strong hyperplanes not yet aligned with the model's provisional axes, we can *see* the suckers as streaks or clusters in the pattern plots taunting us to capture them. To be sure, irresistible item alignments do not always exist in particular applications, and even when they do considerable axis shifting may be needed before they begin to stand out in the pattern plots. But despite their frequent elusiveness, quality hyperplanes take precedence over other desiderata in inductivist search for interpretably distinctive pattern features simply because they grab attention most compellingly, regardless of how we propose thereafter to explain them. (How often we explain them correctly, or make proper allowance for the ease with which they can be artifactual, becomes an issue only after they have been found.)

And this study's disclosure of extant rotation methods' capabilities to detect quality hyperplanes within data that contain them is of course what we can expect to generalize. The factor structure in each of the present datasets is best viewed as an ideal-*cum*-disturbances, where the ideal is an array of data variables with perfect factor complexities at levels more realistic than the independent clusters which have been traditional in simulation studies. In this ideal (before disturbance), perfection consists of each item having purely zero loadings on all factors not declared salient for it in its complexity specification, while the latter are "realistic" in that items are included at all complexity levels up to 3/5ths of the maximum possible 5; specifically, 40% each of levels 2 and 3 with only 20% of items having complexity 1. Despite the higher item complexities, hyperplanes in this ideal structure are so perceptually intense that any inductivist would seize upon them as orgastic revelation if encountered in empirical data. But of course real life is never that tidy; so in most of the datasets studied here this ideal pattern has been degraded by randomly scattering the raw pattern's nonsalient loadings over a bounded interval centered on zero. This changes the perceptually distinctive item alignments from precise planes to fuzzy-edged bands that do not clearly pick out one specific position within them as the inductively best hyperplane therein.

Given that strongest-hyperplanes detection is what this study has undertaken to test, the population factor pattern/covariances here called "source structure" is clearly the proper target for appraising factor recovery from the $W=0$ population covariances (see Appendix B); and it remains so when the factored item covariances are only sample estimates thereof so long as we view analysis of the sample as an attempt to learn the population structure. But are the present "source" factors also the most appropriate targets in the $W>0$ datasets? That depends on how we prefer to represent fuzzy bands by zero-width idealizations thereof. Yet if the number of nonsalient loadings created in production of "source" factors by randomization over interval $0 \pm W$ had been very much larger than the present 14 out of 25 per factor, the resultant source-factor hyperplanes would have been exactly centered in bands of conspicuous item concentration. So the present source-factors' small-sample approximations thereto are surely the most appropriate targets for appraising hyperplane recovery pending consensus on a compelling post-production criterion for strongest hyperplanes in the $W>0$ populations. (I would volunteer $\mathcal{L}_{\mathrm{hybl}}$-optimization as such a criterion did I not fear scornful accusations of cheating.)

That "accuracy" of rotation in this study's report of results refers to detection of strongest hyperplanes is an essential condition on this study's claim to generalizability. The foremost finding here is that all currently established methods of factor rotation are woefully inaccurate in locating high-quality hyperplanes when items that embody them have factor complexities greater than the classic independent-clusters ideal, whereas Hyball rotation finds the hyperplanes of complex items very nicely indeed so long as they are not too diffuse. The practical advice that would seem to follow is that all commercial software for multivariate data analysis should henceforth include some version of Hyball rotation among their factoring options--at the very least access to $\mathcal{L}_{\mathrm{hybl}}$ as an alternative to $\mathcal{L}_{\mathrm{oblm}}$ when serially iterating planar rotations, and preferably some of Hyball's other advanced options as well, notably Spin Search. However, this recommendation presumes that strong hyperplanes are indeed what users often hope to achieve by factor rotation. Otherwise, some more traditional rotation method may well be superior to Hyball for the purpose at hand. But what *are* some rotation goals that at times are worth the price of inferior hyperplanes? Until these are identified and proved attainable more successfully through some variant of the currently standard Orthomax/Promax/Orthoblique/Oblimin than from any version of Hyball, preference for the former is appropriate only when the user's accessible software has not been updated with a routine for $\mathcal{L}_{\mathrm{hybl}}$ optimization.

Insomuch as present findings (*a*) demonstrate that all currently standard rotation methods are decisively obsolete for rotation to best hyperplanes, but (*b*) are not relevant to these methods' comparative merits at achieving other pattern desiderata, review of present results on the varieties of Orthomax/Orthoblique/Oblimin might seem to have little point. Even so, for the benefit of folk who only have time to read summaries, this will head the following gist of what else has been learned from this study about detecting the hyperplanes of items having factor complexities greater than the independent-clusters ideal.

A. Comparisons among established rotation varieties.

1. Kaiser normalization is preferable for both Orthomax and Oblimin, but its benefits are quite small, especially for Oblimin.
2. Among variants of Orthomax, Equamax is clearly superior to Varimax which in turn is even more decisively superior to Quartimax. Variants of Orthomax with γ larger than Equamax setting $\gamma = NF/2$ are perhaps better yet; however, the improvements over Equamax appear trivial, and how well these comparisons generalize beyond the present $NF = 5$ remains untested.
3. Promax does best under the lowest permitted setting, 2, of its powering parameter. And as one would expect, the better the Orthomax pattern to which Promax is applied the better is its result.
4. Direct Oblimin does better when started from Equamax pre-rotation than from the principal-factors extraction pattern, with the difference becoming rather large as its variant parameter γ becomes increasingly negative. However, its Quartimin variant $\gamma = 0$ is so close to optimal that there is little point in fooling with other γ settings.
5. Differences among the best variants of Promax, Orthoblique, and (direct) Oblimin were very small though not quite negligible. Promax was best of all; however, this cannot be expected to generalize to items with lower factor complexities insomuch as theory indicates that Quartimin should be especially adept at locating independent clusters, possibly even as good as Hyball in this highly special case.

B. Sensitivity of rotation results to start position (trapping by local optima).

1. Varimax, Equamax, and presumably all other variants of Orthomax generally converge to essentially the same solution from all orthonormal start positions. This is known to have occasional exceptions, but present evidence encourages us to expect it with high confidence. In contrast, the sensitivity to start position of serially iterated Quartimin rotation, though relatively mild, is still large enough to range over solutions urging significantly different interpretations. And variation of results under randomization of start position is very large indeed for parallel-iterated Quartimin and both iteration styles of Hyball. It is entirely possible that the present datasets provide more opportunity for convergence to optima that are merely local than typical of empirical applications; but it is nevertheless clear that calls of Quartimin or Hyball should be preceded by rotation to a reliably good start position. If Spin search is not feasible, Varimax or better Equamax is strongly recommended.

2. Spin search by a start-sensitive rotation algorithm *Meth*, that is, collecting a goodly number of rotations from random starts which are then ranked on rotation criterion $\mathcal{L}_{\text{meth}}$, effectively enables *Meth* to find the solution that globally optimizes $\mathcal{L}_{\text{meth}}$. But this is a two-edged sword: If $\mathcal{L}_{\text{meth}}$ is an excellent measure of how well a rotation achieves its user's desire, as $\mathcal{L}_{\text{hybl}}$ is of quality hyperplanes, the $\mathcal{L}_{\text{meth}}$-wise best solution found by Spin search may well be substantially superior to what is obtained from any standard start. But if $\mathcal{L}_{\text{meth}}$ is rather poor at its intended task, as $\mathcal{L}_{\text{qmin}}$ is at hyperplane detection, its global optimum found by Spin search may well be inferior to the solution at a local optimum that traps the iteration when started at a good-quality pre-rotation such as Equamax.

3. The individual factors in a rotated pattern generally differ considerably both in their start-sensitivity, which under Spin search is operationally identifiable, and in their recovery accuracy, which is not. Unhappily, the former correlates too poorly with the latter to be usefully diagnostic of it.

C. The glory of suboptimal factor rotation revisited.

1. In Rozeboom, 1993, I argued--mainly on theoretical grounds--that start sensitivity is more beneficial than detrimental in a routine for factor rotation, insomuch as some of the local

minima which can trap it may well realize the user's rotation goal better than does the solution at $\mathcal{L}_{meth}$'s global optimum. (Of course, exploitation of this prospect requires Spin search. But that should be no problem: If mainstream multivariate software proves slow to provide this, you can easily acquire Hyball.) Present results show that this prospect is indeed a practical reality. Spin collections from all the start-sensitive rotation methods tested here included solutions that were appreciably more accurate in source recovery than the solution ranked tops on criterion; and for Hyball this gain was quite substantial for patterns in which diffuse hyperplanes largely defeated Hyball's special competence at detection thereof. Admittedly, my comments a few paragraphs ago on idealizing fuzzy hyperplanes may have left you with doubts whether this study's official source patterns are truly the best targets for appraising solution accuracy in the $W > 0$ datasets. But that doesn't really matter: The essential point, demonstrated here by superior source recovery at some of the merely-local optima, is that the Cream of Spin search may well contain solutions better than Rank 1 for one purpose or another. How well the present source patterns can be recovered is certainly of legitimate interest, insomuch as they are as much like genuine causes of factored variables as artificial data can probably simulate; and if their hyperplanes are not in perfect agreement with the population hyperplanes demarked as strongest by one or another analytic measure thereof, that is only what we should expect from the hyperplanes of empirical causes as well. Whether the information we hope to get from factor rotation addresses causes or something else of which quality hyperplanes are imperfectly diagnostic, subjective appraisal of the different solutions proffered by filtered Spin Cream enables us to choose final axis positionings in which analytic ratings of hyperplane strength are tempered by judgments of interpretive quality.

Acknowledgment

My insistence that Hyball's demonstrable performance quality makes our current repertoire of "established," "mainstream," or "standard" rotation methods largely obsolete should in no way be taken to suggest that Hyball's approach is lacking in tradition. As you are undoubtedly aware, analytic rotation to optimize a strength-of-hyperplanes criterion was pioneered by Cattell & Muerle (1960) and Eber (1966) using, however, a pass/fail hyperplane count whose sensitivity has proved wanting. The more powerful appraisal of hyperplane strength by a graded measure of item fit that effectively ignores outliers was first conceived by Katz & Rohlf (1974) in the form of an exponential function very similar in character to many in the class from which Hyball control parameters allow selection. The Katz-Rohlf function has not been included among Hyball's options because it is much slower than and not quite so accurate in source recovery as the best $\mathcal{L}_{\text{hybl}}$ variants. But its proposal was a major advance in the theory of hyperplane detection that well deserves respectful recognition.

APPENDIX A. RMS Difference vs. Divergence measures of vector similarity.

Let **x** and **y** be two order-n column vectors. Then their root-mean-square difference, RMS(**x**-**y**), and congruence Divergence, Div(**x**,**y**), are by definition respectively

$$\text{RMS}(\mathbf{x}-\mathbf{y}) = [n^{-1}(\mathbf{x}-\mathbf{y})'(\mathbf{x}-\mathbf{y})]^{\frac{1}{2}} = [n^{-1}(\mathbf{x}'\mathbf{x} + \mathbf{y}'\mathbf{y} - 2\mathbf{x}'\mathbf{y})]^{\frac{1}{2}}$$

$$\text{Div}(\mathbf{x},\mathbf{y}) = \cos^{-1}(|\mathbf{x}'\mathbf{y}(\mathbf{x}'\mathbf{x}\mathbf{y}'\mathbf{y})^{-\frac{1}{2}}|) = \cos^{-1}(|r_{xy}|), \quad r_{xy} =_{\text{def}} \mathbf{x}'\mathbf{y}/(\mathbf{x}'\mathbf{x}\mathbf{y}'\mathbf{y})^{\frac{1}{2}}$$

where r_{xy} is the uncentered correlation between **x** and **y**, that is, the cosine of their vectorial angle. Under two side stipulations on **x** and **y**, there is a remarkably simple relation between these two similarity measures.

Specifically, presume (1) that **x** and **y** have the same orientation, that is $r_{xy} \geq 0$, and (2) that **x** and **y** have the same euclidian norm s, that is $\mathbf{x}'\mathbf{x} = s^2 = \mathbf{y}'\mathbf{y}$. (When **x** is, or estimates, some column of a conventionally scaled factor pattern also estimated by **y**, (1) can be assured by stipulation while (2) should also be decently approximated.) Then $r_{xy} = \mathbf{x}'\mathbf{y}/s^2$ while $\text{RMS}(\mathbf{x}-\mathbf{y}) = [n^{-1}s^2(2-2\mathbf{x}'\mathbf{y}/s^2)]^{\frac{1}{2}}$ $= [(n^{-1}s^2)\cdot 2(1-r_{xy})]^{\frac{1}{2}} = \text{RMS}(\mathbf{x})\cdot\sqrt{2}\cdot[1-\cos(\text{Div}(\mathbf{x},\mathbf{y}))]^{\frac{1}{2}}$, since $s^2 = \mathbf{x}'\mathbf{x} = n\cdot\text{RMS}(\mathbf{x})^2$ by (2) and $r_{xy} = |r_{xy}|$ $= \cos(\text{Div}(\mathbf{x},\mathbf{y}))$ by (1). Now, for any angle α in range 0° - 90°, $[1-\cos(\alpha)]^{\frac{1}{2}} \approx .012\alpha$ to an extremely close approximation whose error is on the order of .005 when $\alpha < 60°$ and for larger α increases only to .080 even at $\alpha = 90°$. (I discovered this by brute-force plotting of paired values. It undoubtedly has a good analytic proof which, however, has eluded me.) Hence given (1) and (2), since $.012 \cdot \sqrt{2} \approx .017$,

$$\mathrm{RMS}(\mathbf{x}\text{-}\mathbf{y}) \approx .017 \cdot \mathrm{RMS}(\mathbf{x}) \cdot \mathrm{Div}(\mathbf{x},\mathbf{y}) \ .$$

When **x** is a column of a conventionally scaled source pattern or decent estimate thereof, we can bracket RMS(**x**) with plausible bounds as follows: Let the loading magnitudes in **x** be partitioned into two groups, nonsalient (small) and salient (not so small). Then $\mathrm{RMS}(\mathbf{x})^2$ is the sum over groups g = <salient, nonsalient> of $p_g(\mu_g^2 + \sigma_g^2)$, where p_g is the proportion of loadings, and μ_g and σ_g their magnitudes' mean and SD, in group g. If we put the salient/nonsalient cut around .25 or .30, it seems reasonable to expect $p_{salient}$ generally between .10 and .40 for middle-sized patterns (or perhaps a bit less for quite large ones), $\mu_{salient}$ between .5 and .6, $\sigma_{salient}$ between .10 and .15, $\mu_{nonsalient}$ between .10 and .15, and $\sigma_{nonsalient}$ between .05 and .10, yielding a rule-of-thumb anticipated range for RMS(**x**) of .163 to .415. This in turn implies RMS(**x**-**y**) between .00028 and .0071 times Div(**x**,**y**). Or more simply, multiplying pattern elements by 100 to reflect intuitive disregard of decimal points in two-place pattern loadings, 100•RMS(**x**-**y**) should generally be from three-tenths to seven-tenths as large as Div(**x**,**y**).

APPENDIX B. Recovery accuracies uncontaminated by sampling error.

The simulation data analyzed in this study have been corrupted from the structure of their framework ideal in two rather different ways: First, the perfect ($W = 0$) hyperplanes in the raw source patterns were disturbed by varied severities of hyperplane noise. And second, constructing each analyzed dataset as a random selection of only 400 records from population introduced sampling errors into the factored item covariances that also degraded the best possible source recovery. Unlike the first type of impurity, which is structural, the second can be eliminated in practice (even if only at considerable cost) by collecting data from very large samples. So it is also of interest to see how much of the recovery inaccuracy observed in this study is due to structural imperfection detached from sampling error.

To examine this, the SPINTEST runs whose accuracy reports are shown in Table 3 and Figure 1 were repeated with their sampled item covariances replaced by population values, yielding the Rank-1 source-recovery accuracies for Quartimin(S) and Hyball(P), together with Best on Hyball(P), plotted in Figure 2 with solid lines. (For ease of comparison, the corresponding accuracies-in-sample from Table 3 are also shown there with dashed lines. Note that Figure 2 omits comparisons on DivP to make room for the MaxC comparisons squeezed out of Figure 1.)

Figure 2 about here

You will observe that for Hyball, source recovery from the population data is nearly perfect on all *Sims*, even MaxP and MaxC, at noise level $W=0$, and more generally improves upon the $NS=400$ results by an amount roughly constant over all W-levels except $W=.20$, at which the gain sharply deteriorates. That elimination of sampling error cannot much improve Hyball detection of hyperplanes weaker than its discrimination threshold is not surprising. Rather more surprising is how much this improves detection of difficult hyperplanes on which Hyball can get some purchase. In particular, the MaxP and MaxC errors, which in Hyball recovery from the samples are large enough to be interpretively disturbing even at W-levels 0 and .05, have subsided in population recovery to tolerable at $W \leq .10$ and are substantially tamed even if still troublesome at $W=.15$. There is a clear practical admonition in this: In applications where precise positioning of factor axes is important, data collection from very large samples may indeed be cost-effective.

In contrast to Hyball's pronounced gain from elimination of sampling error, the other rotation methods studied here benefited considerably less from this. In particular, MaxP and MaxC remain interpretively destructive in their solutions at all levels of hyperplane noise.

APPENDIX C. Effects of sampling error and hyperplane noise on pattern loadings.

Because Hyball's performance in this study has been so strongly influenced both by W-disturbance and sampling error, you should be interested in how these are manifested in the scatter of source-factor loadings. Table 6 shows for each W-level (*a*) the distribution of loading magnitudes over all factors in all 20 population source patterns at that W, with the subdistribution of loadings generated as nonsalient separated from that of the salients; and (*b*) the same breakdown of loading distributions in the sample data's procrustes approximation to source, this being essentially the closest match to the source pattern attainable in the sample.

Table 6 about here

Table 6 does not, however, tell how these loadings were jointly scattered in the factor planes, lacking which information the Table 6 distributions do not make clear why W had so strong an effect in this study. Although it is impractical to show representative samples of these planes here, this can be partly accomplished by advising you that in each plane of every both source pattern and its procrustes approximation, four items were constructed as salient on both factors, 14 as salient on just one (divided equally between the two), and four were salient on neither. Also, the proportion of negative nonsalient loadings in each plane deviated by chance from 50% with sampling error .13. The incidence of negative salient loadings was about 15% over all patterns, broken down with 0%, 10%, 20%, and 30% negatives in five datasets each at each W-level. For contemplating how these planar point scatters make mischief for hyperplane recovery, it does no harm to treat all the salient loadings as positive. (This is mainly because none of the rotation methods tested here is sensitive to item orientation. The nonsalients' zero centering and independence of the salients are also relevant, but fine details on that aren't worth your bother.)

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Rozeboom, W. W. (199?). More than you may want to know about source recovery by common-factoring. (Submitted to *MBR*).

FOOTNOTES

1. [Question for referees: Does anyone care for clarification of this?]

2. A later side study found differences in results under these convergence alternatives to be negligible.

3. At the time, I had neither technical competence in this matter nor any desire to play in its major league. But I needed a special rotation feature--the ability to hold selected factor subspaces invariant during the rotation--that extant rotation programs did not provide; and from there, as so often befalls, one thing led to another.

4. $[\mathbf{W}]_{jk}$ is also zero if subspace constraints forbid movement of f_j in this plane. This is an important Hyball option that does not, however, arise in the present study.

5. I generally choose δ to be .5 or .6. Small δ needlessly retards solution speed, while the more closely δ approaches 1 the greater the risk of nonconvergence, especially in STEP mode; but otherwise, results appear to be highly insensitive to variation in δ. In Table 1 and Hyball screen messages, δ is called "DF".

6. I have also generalized this to identifying the loading difference whose percentile rank within the unsigned differences over the full array of loading comparisons is a stipulated value P. Later, I shall mention an interesting finding for $P = 95$.

7. If you view the notion of "targeted source structure" as ontologically dubious in empirical applications, take this to envision whatever axis positioning you most strongly desire to be your factor inquiry's payoff.

8. Because the Min and Max pattern-column divergences in Table 4 are in all $W \geq .15$ entries nearly equal in their divergence from the corresponding means, with Max a tad the larger, the mean divergences are also close to though a tad larger than the mean divergences after exclusion of the best and worst matched factors.

9. OK, so you knew that already. No harm in noting it again.

TABLE 1

Source-recovery accuracies from standard start-positions of all studied rotation variants over 100 similation datasets. Variant index δδ selected $\gamma = \delta\delta$ for Orthomax, $\gamma = -\delta\delta$ for Oblimin, and Hyball versions as follows:

δδ	BH .15	BH .20	BH .25	BH 2.0
WSAL .0	.0	.5	1.0	
WSAL 1.0	1.5	2.0	2.5	
WSAL 2.0	3.0	3.5		4.0*

*** Also JA = 0 to complete Quartimin approximation. Other Hyball parameter settings shared by all were SCAN mode, < JA, JB, CV > = < 1, 2, 1.0 >, planar search window ± 60.0°, and convergence controls < DF, CLOSE, IMAX > = < .50, 1.0, 60 >.**

Promax(K) is power-2 Promax based on variant-δδ Equamax.
Oblimin/Hyball suffixes (P) and (S) respectively demark Parallel and Serial iteration.

A. Mean (SD) RMS DIFFERENCE from the source PATTERN over all hyperplane-noise levels

Extraction-axes start, NORM = 0

δδ	Orthomax	Promax(2)	Promax(4)	Promax(6)	Promax(8)	Oblimin(P)	Oblimin(S)	Hyball(P)	Hyball(S)
.00	.233 (.04)	.217 (.05)	.212 (.05)	.218 (.05)	.225 (.05)	.178 (.05)	.156 (.05)	.253 (.09)	.254 (.09)
.50	.220 (.04)	.202 (.05)	.200 (.05)	.208 (.05)	.216 (.05)	.202 (.06)	.155 (.05)	.229 (.09)	.206 (.10)
1.00	.201 (.04)	.182 (.05)	.184 (.05)	.195 (.05)	.205 (.05)	.230 (.06)	.160 (.05)	.196 (.09)	.158 (.09)
1.50	.183 (.05)	.163 (.06)	.167 (.05)	.180 (.05)	.191 (.05)	.258 (.06)	.161 (.05)	.203 (.09)	.177 (.09)
2.00	.174 (.05)	.153 (.05)	.159 (.05)	.172 (.05)	.184 (.05)	.279 (.05)	.166 (.05)	.179 (.08)	.162 (.09)
2.50	.172 (.05)	.150 (.05)	.156 (.05)	.169 (.05)	.181 (.05)	.292 (.03)	.170 (.05)	.156 (.08)	.146 (.08)
3.00	.170 (.04)	.148 (.05)	.153 (.05)	.167 (.05)	.179 (.05)	.294 (.03)	.176 (.05)	.174 (.08)	.173 (.08)
3.50	.169 (.04)	.147 (.05)	.151 (.05)	.165 (.05)	.177 (.05)	.297 (.03)	.182 (.05)	.161 (.08)	.153 (.07)
4.00	.170 (.04)	.147 (.05)	.151 (.05)	.164 (.05)	.177 (.05)	.302 (.02)	.192 (.06)	.174 (.05)	.169 (.05)

Extraction-axes start, NORM = 1

δδ	Orthomax	Promax(2)	Promax(4)	Promax(6)	Promax(8)	Oblimin(P)	Oblimin(S)	Hyball(P)	Hyball(S)
.00	.224 (.04)	.208 (.04)	.203 (.05)	.209 (.05)	.217 (.05)	.181 (.06)	.145 (.04)	.276 (.08)	.276 (.08)
.50	.207 (.04)	.187 (.05)	.185 (.05)	.192 (.05)	.201 (.05)	.214 (.06)	.151 (.05)	.259 (.09)	.248 (.09)
1.00	.183 (.04)	.164 (.05)	.166 (.05)	.176 (.05)	.187 (.05)	.245 (.06)	.148 (.04)	.236 (.10)	.211 (.09)
1.50	.166 (.04)	.145 (.04)	.149 (.05)	.161 (.04)	.172 (.04)	.273 (.05)	.155 (.04)	.218 (.09)	.202 (.09)
2.00	.160 (.04)	.138 (.04)	.142 (.04)	.154 (.04)	.166 (.04)	.287 (.04)	.167 (.05)	.194 (.09)	.181 (.09)
2.50	.159 (.04)	.136 (.04)	.141 (.04)	.153 (.04)	.165 (.04)	.292 (.04)	.176 (.05)	.181 (.08)	.158 (.08)
3.00	.156 (.03)	.133 (.04)	.137 (.04)	.150 (.04)	.162 (.04)	.295 (.03)	.185 (.05)	.174 (.08)	.168 (.08)
3.50	.156 (.03)	.133 (.04)	.136 (.04)	.150 (.04)	.162 (.04)	.298 (.03)	.188 (.05)	.170 (.08)	.152 (.08)
4.00	.157 (.03)	.134 (.04)	.137 (.04)	.150 (.04)	.162 (.04)	.302 (.02)	.195 (.05)	.174 (.05)	.162 (.04)

Equimax start, NORM = 0

δδ	Orthomax	Promax(2)	Promax(4)	Promax(6)	Promax(8)	Oblimin(P)	Oblimin(S)	Hyball(P)	Hyball(S)
.00	.231 (.04)	.213 (.04)	.208 (.05)	.213 (.05)	.219 (.05)	.137 (.04)	.148 (.04)	.115 (.06)	.127 (.07)
.50	.216 (.04)	.198 (.05)	.195 (.05)	.203 (.05)	.212 (.05)	.135 (.04)	.142 (.04)	.112 (.06)	.123 (.07)
1.00	.198 (.05)	.178 (.05)	.180 (.05)	.191 (.05)	.202 (.05)	.135 (.04)	.138 (.04)	.111 (.05)	.116 (.06)
1.50	.182 (.05)	.162 (.06)	.166 (.05)	.179 (.05)	.190 (.05)	.135 (.04)	.138 (.04)	.111 (.06)	.118 (.07)
2.00	.172 (.04)	.150 (.05)	.156 (.05)	.170 (.05)	.182 (.05)	.136 (.04)	.137 (.04)	.109 (.06)	.119 (.07)
2.50	.169 (.04)	.147 (.05)	.153 (.05)	.167 (.05)	.179 (.05)	.136 (.04)	.137 (.04)	.112 (.05)	.117 (.06)
3.00	.168 (.04)	.146 (.05)	.151 (.05)	.165 (.05)	.178 (.05)	.136 (.04)	.138 (.04)	.121 (.06)	.128 (.06)
3.50	.169 (.04)	.146 (.05)	.150 (.05)	.164 (.05)	.176 (.05)	.136 (.04)	.138 (.04)	.121 (.06)	.127 (.07)
4.00	.168 (.04)	.146 (.05)	.150 (.05)	.163 (.04)	.176 (.05)	.136 (.04)	.138 (.04)	.155 (.04)	.162 (.04)

Equimax start, NORM = 1

δδ	Orthomax	Promax(2)	Promax(4)	Promax(6)	Promax(8)	Oblimin(P)	Oblimin(S)	Hyball(P)	Hyball(S)
.00	.221 (.04)	.203 (.04)	.199 (.04)	.204 (.05)	.213 (.05)	.136 (.04)	.139 (.04)	.121 (.07)	.140 (.08)
.50	.206 (.04)	.186 (.05)	.184 (.05)	.191 (.05)	.200 (.05)	.136 (.04)	.137 (.04)	.116 (.07)	.126 (.07)
1.00	.182 (.04)	.162 (.05)	.164 (.05)	.174 (.05)	.185 (.05)	.136 (.04)	.136 (.04)	.114 (.06)	.122 (.07)
1.50	.165 (.04)	.144 (.04)	.148 (.04)	.160 (.04)	.172 (.04)	.136 (.04)	.136 (.04)	.115 (.06)	.125 (.07)
2.00	.160 (.04)	.138 (.04)	.142 (.04)	.154 (.04)	.166 (.04)	.136 (.04)	.137 (.04)	.111 (.06)	.116 (.07)
2.50	.159 (.04)	.136 (.04)	.141 (.04)	.153 (.04)	.165 (.04)	.136 (.04)	.137 (.04)	.110 (.06)	.115 (.07)
3.00	.158 (.04)	.136 (.04)	.140 (.04)	.153 (.04)	.164 (.04)	.136 (.04)	.137 (.04)	.122 (.06)	.121 (.07)
3.50	.158 (.04)	.135 (.04)	.138 (.04)	.151 (.04)	.163 (.04)	.136 (.04)	.138 (.04)	.116 (.06)	.118 (.06)
4.00	.158 (.04)	.135 (.04)	.138 (.04)	.151 (.04)	.163 (.04)	.136 (.04)	.138 (.04)	.150 (.04)	.156 (.04)

B. Mean (SD) congruence DIVERGENCE from the source PATTERN over all hyperplane-noise levels

Extraction-axes start, NORM = 0

δδ	Orthomax	Promax(2)	Promax(4)	Promax(6)	Promax(8)	Oblimin(P)	Oblimin(S)	Hyball(P)	Hyball(S)
.00	35.1 (6.2)	32.6 (7.3)	31.9 (7.2)	32.3 (7.0)	32.8 (6.9)	27.6 (7.9)	24.1 (7.1)	32.1 (11.)	31.5 (10.)
.50	32.9 (6.8)	30.3 (7.8)	30.1 (7.6)	30.9 (7.4)	31.5 (7.3)	31.6 (9.8)	24.1 (7.5)	29.7 (11.)	26.2 (11.)
1.00	30.1 (7.0)	27.2 (8.1)	27.7 (7.6)	29.0 (7.3)	30.0 (7.1)	36.4 (9.6)	24.9 (7.8)	26.2 (11.)	21.3 (10.)
1.50	27.7 (6.9)	24.5 (8.2)	25.4 (7.8)	27.1 (7.4)	28.2 (7.2)	40.9 (9.1)	25.2 (7.4)	27.4 (11.)	24.1 (12.)
2.00	26.7 (6.7)	23.2 (7.8)	24.2 (7.5)	26.0 (7.2)	27.3 (7.1)	44.3 (7.6)	26.0 (7.7)	25.0 (11.)	22.0 (11.)
2.50	26.5 (6.7)	22.9 (7.8)	23.9 (7.4)	25.6 (7.1)	26.9 (6.9)	46.3 (5.1)	26.7 (7.7)	22.3 (10.)	20.9 (11.)
3.00	26.3 (6.5)	22.7 (7.5)	23.6 (7.2)	25.4 (6.9)	26.7 (6.8)	46.5 (4.8)	27.7 (7.6)	24.7 (11.)	24.1 (11.)
3.50	26.2 (6.5)	22.5 (7.5)	23.4 (7.1)	25.2 (6.9)	26.5 (6.7)	47.0 (4.3)	28.7 (8.1)	22.9 (10.)	21.7 (10.)
4.00	26.3 (6.5)	22.6 (7.5)	23.4 (7.1)	25.1 (6.9)	26.4 (6.7)	47.7 (3.6)	30.3 (8.9)	26.7 (7.1)	26.1 (7.3)

Extraction-axes start, NORM = 1

δδ	Orthomax	Promax(2)	Promax(4)	Promax(6)	Promax(8)	Oblimin(P)	Oblimin(S)	Hyball(P)	Hyball(S)
.00	33.4 (5.8)	30.7 (6.6)	30.1 (6.7)	30.5 (6.5)	31.1 (6.4)	27.9 (8.3)	22.4 (6.5)	34.9 (10.)	33.7 (9.2)
.50	30.3 (6.0)	27.6 (7.0)	27.6 (6.9)	28.4 (6.6)	29.2 (6.5)	33.4 (9.8)	23.4 (7.0)	33.0 (10.)	30.5 (11.)
1.00	27.2 (6.0)	24.2 (7.2)	24.8 (7.0)	26.1 (6.7)	27.2 (6.5)	38.4 (9.2)	23.2 (6.5)	30.0 (12.)	26.8 (11.)
1.50	25.4 (5.6)	22.0 (6.7)	22.9 (6.6)	24.5 (6.3)	25.7 (6.2)	42.8 (7.8)	24.3 (6.8)	29.4 (12.)	27.1 (11.)
2.00	24.7 (5.3)	21.0 (6.3)	21.9 (6.0)	23.5 (5.7)	24.8 (5.6)	45.2 (5.9)	26.4 (7.5)	26.7 (12.)	24.4 (11.)
2.50	24.6 (5.2)	20.8 (6.2)	21.7 (5.9)	23.4 (5.6)	24.7 (5.6)	46.0 (5.3)	27.7 (7.9)	25.0 (11.)	22.2 (11.)
3.00	24.2 (4.7)	20.4 (5.6)	21.2 (5.4)	22.9 (5.2)	24.3 (5.2)	46.5 (4.9)	29.1 (7.6)	24.6 (11.)	23.4 (11.)
3.50	24.2 (4.8)	20.4 (5.6)	21.2 (5.4)	22.9 (5.2)	24.3 (5.2)	47.0 (4.7)	29.8 (7.7)	23.7 (11.)	21.6 (11.)
4.00	24.4 (5.0)	20.6 (5.9)	21.3 (5.6)	23.0 (5.4)	24.4 (5.3)	47.6 (3.6)	31.0 (7.8)	26.9 (7.4)	25.1 (6.6)

Equimax start, NORM = 0

δδ	Orthomax	Promax(2)	Promax(4)	Promax(6)	Promax(8)	Oblimin(P)	Oblimin(S)	Hyball(P)	Hyball(S)
.00	34.6 (6.3)	32.0 (7.2)	31.2 (7.2)	31.6 (7.1)	32.1 (7.0)	21.2 (5.9)	22.9 (7.1)	16.4 (8.3)	17.3 (9.1)
.50	32.2 (6.5)	29.5 (7.6)	29.4 (7.4)	30.3 (7.2)	31.0 (7.1)	20.9 (5.9)	22.0 (6.4)	16.1 (8.1)	17.2 (9.1)
1.00	29.4 (7.0)	26.5 (8.1)	27.1 (7.7)	28.5 (7.3)	29.6 (7.2)	21.0 (5.9)	21.5 (5.9)	16.3 (7.1)	16.7 (7.8)
1.50	27.6 (6.9)	24.3 (8.2)	25.2 (7.7)	26.9 (7.3)	28.1 (7.2)	21.0 (5.9)	21.5 (5.8)	16.2 (8.4)	17.1 (9.1)
2.00	26.3 (6.4)	22.8 (7.5)	23.8 (7.2)	25.7 (6.9)	27.0 (6.8)	21.0 (5.9)	21.4 (5.8)	16.0 (7.8)	17.2 (8.8)
2.50	26.1 (6.2)	22.4 (7.3)	23.4 (7.0)	25.2 (6.7)	26.6 (6.6)	21.0 (5.9)	21.4 (5.8)	16.6 (7.4)	17.2 (8.0)
3.00	26.1 (6.3)	22.3 (7.2)	23.3 (6.8)	25.1 (6.6)	26.4 (6.5)	21.0 (5.9)	21.4 (5.8)	17.6 (8.8)	18.2 (8.7)
3.50	26.1 (6.3)	22.3 (7.1)	23.2 (6.8)	25.0 (6.6)	26.3 (6.5)	21.0 (5.9)	21.5 (5.8)	17.7 (8.4)	18.3 (9.0)
4.00	26.1 (6.3)	22.4 (7.1)	23.2 (6.8)	25.0 (6.6)	26.3 (6.5)	21.1 (5.9)	21.5 (5.8)	23.9 (6.2)	24.9 (6.4)

Equimax start, NORM = 1

δδ	Orthomax	Promax(2)	Promax(4)	Promax(6)	Promax(8)	Oblimin(P)	Oblimin(S)	Hyball(P)	Hyball(S)
.00	32.9 (5.8)	30.1 (6.7)	29.5 (6.7)	30.0 (6.6)	30.6 (6.6)	21.0 (5.9)	21.6 (6.2)	17.2 (8.6)	19.1 (10.)
.50	30.2 (6.1)	27.5 (7.0)	27.4 (6.9)	28.3 (6.6)	29.2 (6.5)	21.0 (5.9)	21.3 (5.9)	16.5 (8.7)	17.2 (8.8)
1.00	27.1 (6.0)	24.1 (7.2)	24.7 (6.9)	26.0 (6.6)	27.1 (6.4)	21.0 (5.9)	21.2 (5.8)	16.4 (8.2)	16.9 (8.5)
1.50	25.3 (5.6)	21.8 (6.7)	22.7 (6.5)	24.3 (6.3)	25.6 (6.2)	21.0 (5.9)	21.2 (5.8)	16.7 (8.7)	17.7 (9.8)
2.00	24.7 (5.3)	21.0 (6.3)	21.9 (6.0)	23.5 (5.7)	24.8 (5.7)	21.0 (5.9)	21.3 (5.8)	16.3 (8.3)	16.8 (9.0)
2.50	24.6 (5.2)	20.8 (6.2)	21.7 (5.9)	23.4 (5.6)	24.7 (5.6)	21.1 (5.9)	21.4 (5.8)	16.3 (8.2)	16.8 (9.1)
3.00	24.5 (5.1)	20.8 (6.1)	21.6 (5.9)	23.3 (5.6)	24.6 (5.5)	21.1 (5.9)	21.4 (5.8)	17.8 (8.9)	17.7 (9.2)
3.50	24.4 (5.1)	20.7 (6.0)	21.4 (5.7)	23.1 (5.5)	24.4 (5.4)	21.1 (5.9)	21.5 (5.7)	17.2 (8.8)	17.3 (8.7)
4.00	24.6 (5.2)	20.8 (6.1)	21.5 (5.8)	23.2 (5.6)	24.5 (5.5)	21.1 (5.9)	21.5 (5.7)	23.4 (6.2)	24.2 (6.3)

C. **Mean (SD) RMS DIFFERENCE from the source COVARIANCES over all hyperplane-noise levels**

Extraction-axes start, NORM = 0

δδ	Orthomax	Promax(2)	Promax(4)	Promax(6)	Promax(8)	Oblimin(P)	Oblimin(S)	Hyball(P)	Hyball(S)
.00	.283 (.04)	.243 (.05)	.267 (.06)	.296 (.07)	.317 (.08)	.217 (.06)	.208 (.05)	.343 (.12)	.356 (.13)
.50	.283 (.04)	.235 (.05)	.263 (.06)	.296 (.07)	.319 (.08)	.227 (.05)	.206 (.05)	.325 (.13)	.301 (.15)
1.00	.283 (.04)	.222 (.05)	.251 (.06)	.289 (.08)	.316 (.09)	.244 (.05)	.204 (.04)	.273 (.12)	.227 (.13)
1.50	.283 (.04)	.210 (.05)	.247 (.06)	.291 (.08)	.320 (.09)	.258 (.05)	.205 (.04)	.286 (.13)	.259 (.15)
2.00	.283 (.04)	.202 (.05)	.238 (.06)	.281 (.08)	.306 (.09)	.270 (.05)	.209 (.04)	.257 (.13)	.237 (.14)
2.50	.283 (.04)	.199 (.05)	.236 (.05)	.275 (.07)	.303 (.09)	.276 (.04)	.208 (.04)	.236 (.11)	.219 (.12)
3.00	.283 (.04)	.197 (.05)	.233 (.05)	.272 (.07)	.296 (.08)	.277 (.04)	.210 (.04)	.255 (.12)	.260 (.12)
3.50	.283 (.04)	.197 (.05)	.231 (.05)	.273 (.07)	.298 (.08)	.279 (.04)	.213 (.04)	.243 (.11)	.232 (.12)
4.00	.283 (.04)	.197 (.04)	.230 (.05)	.272 (.07)	.298 (.08)	.282 (.04)	.220 (.04)	.257 (.07)	.263 (.07)

Extraction-axes start, NORM = 1

δδ	Orthomax	Promax(2)	Promax(4)	Promax(6)	Promax(8)	Oblimin(P)	Oblimin(S)	Hyball(P)	Hyball(S)
.00	.283 (.04)	.245 (.05)	.264 (.07)	.295 (.08)	.322 (.09)	.215 (.05)	.195 (.05)	.370 (.12)	.391 (.12)
.50	.283 (.04)	.232 (.05)	.251 (.06)	.287 (.08)	.314 (.09)	.231 (.05)	.200 (.05)	.350 (.13)	.354 (.14)
1.00	.283 (.04)	.213 (.05)	.243 (.06)	.284 (.08)	.315 (.09)	.248 (.05)	.197 (.04)	.322 (.14)	.311 (.15)
1.50	.283 (.04)	.201 (.05)	.233 (.06)	.273 (.07)	.303 (.09)	.265 (.05)	.202 (.05)	.298 (.13)	.305 (.15)
2.00	.283 (.04)	.193 (.04)	.225 (.05)	.267 (.07)	.297 (.08)	.271 (.05)	.209 (.05)	.281 (.13)	.276 (.15)
2.50	.283 (.04)	.193 (.04)	.224 (.05)	.267 (.07)	.296 (.08)	.273 (.05)	.213 (.05)	.256 (.11)	.234 (.13)
3.00	.283 (.04)	.192 (.04)	.223 (.05)	.262 (.07)	.291 (.08)	.277 (.04)	.223 (.06)	.250 (.12)	.253 (.13)
3.50	.283 (.04)	.191 (.04)	.222 (.05)	.262 (.07)	.291 (.08)	.279 (.04)	.223 (.05)	.251 (.12)	.234 (.13)
4.00	.283 (.04)	.191 (.04)	.222 (.05)	.262 (.07)	.290 (.08)	.281 (.04)	.225 (.05)	.257 (.07)	.247 (.06)

Equimax start, NORM = 0

δδ	Orthomax	Promax(2)	Promax(4)	Promax(6)	Promax(8)	Oblimin(P)	Oblimin(S)	Hyball(P)	Hyball(S)
.00	.283 (.04)	.243 (.05)	.263 (.06)	.291 (.07)	.312 (.08)	.196 (.04)	.205 (.05)	.178 (.11)	.194 (.12)
.50	.283 (.04)	.234 (.05)	.262 (.06)	.293 (.07)	.316 (.08)	.194 (.04)	.197 (.04)	.175 (.11)	.184 (.12)
1.00	.283 (.04)	.221 (.05)	.252 (.06)	.291 (.07)	.319 (.09)	.193 (.04)	.194 (.04)	.176 (.09)	.185 (.10)
1.50	.283 (.04)	.209 (.05)	.246 (.06)	.287 (.07)	.316 (.09)	.194 (.04)	.194 (.04)	.173 (.10)	.183 (.11)
2.00	.283 (.04)	.201 (.05)	.236 (.06)	.276 (.07)	.302 (.09)	.194 (.04)	.193 (.04)	.171 (.10)	.181 (.11)
2.50	.283 (.04)	.198 (.05)	.234 (.05)	.274 (.07)	.300 (.09)	.194 (.04)	.192 (.04)	.176 (.09)	.182 (.09)
3.00	.283 (.04)	.197 (.05)	.233 (.05)	.272 (.07)	.298 (.08)	.194 (.04)	.192 (.04)	.194 (.11)	.203 (.11)
3.50	.283 (.04)	.197 (.04)	.230 (.05)	.271 (.07)	.296 (.08)	.193 (.04)	.192 (.04)	.189 (.10)	.198 (.10)
4.00	.283 (.04)	.197 (.05)	.231 (.05)	.270 (.07)	.296 (.08)	.193 (.04)	.192 (.04)	.255 (.07)	.256 (.06)

Equimax start, NORM = 1

δδ	Orthomax	Promax(2)	Promax(4)	Promax(6)	Promax(8)	Oblimin(P)	Oblimin(S)	Hyball(P)	Hyball(S)
.00	.283 (.04)	.242 (.05)	.258 (.07)	.286 (.08)	.312 (.08)	.192 (.04)	.191 (.04)	.194 (.12)	.215 (.12)
.50	.283 (.04)	.231 (.05)	.251 (.06)	.288 (.08)	.316 (.09)	.192 (.04)	.190 (.04)	.175 (.10)	.193 (.12)
1.00	.283 (.04)	.213 (.05)	.243 (.06)	.285 (.08)	.316 (.09)	.192 (.04)	.190 (.04)	.176 (.10)	.189 (.12)
1.50	.283 (.04)	.200 (.05)	.232 (.06)	.273 (.07)	.302 (.08)	.193 (.04)	.190 (.04)	.179 (.11)	.191 (.12)
2.00	.283 (.04)	.193 (.04)	.225 (.05)	.267 (.07)	.298 (.08)	.193 (.04)	.190 (.04)	.172 (.10)	.173 (.11)
2.50	.283 (.04)	.193 (.04)	.224 (.05)	.267 (.07)	.296 (.08)	.193 (.04)	.191 (.04)	.170 (.10)	.176 (.11)
3.00	.283 (.04)	.192 (.04)	.226 (.06)	.264 (.07)	.293 (.08)	.193 (.04)	.191 (.04)	.193 (.11)	.195 (.11)
3.50	.283 (.04)	.193 (.04)	.223 (.05)	.264 (.07)	.293 (.08)	.193 (.04)	.191 (.04)	.179 (.10)	.182 (.10)
4.00	.283 (.04)	.192 (.04)	.223 (.05)	.263 (.07)	.293 (.08)	.193 (.04)	.191 (.04)	.245 (.06)	.245 (.06)

D. Breakdown by hyperplane-noise level W of selected pattern-recovery measures for selected method variants under NORM = 1 from both Equamax start and Varimax start. For comparison to Spin results in Table 3, the most relevant portions of this are δδ = .00 for OblmS (serially iterated Oblimin) and δδ = 2.0 for HyblP (parallel iterated Hyball). Columns "…/Eqmx" are results from Equamax start, columns "…/Vmx" are from Varimax start.

RMS DIFFERENCE from the source PATTERN

δδ	W	OblmS/Eqmx	OblmS/Vmx	HyblP/Eqmx	HyblP/Vmx
.0	.00	.116 (.02)	.122 (.03)	.071 (.03)	.108 (.05)
.0	.05	.117 (.03)	.119 (.03)	.082 (.05)	.096 (.05)
.0	.10	.134 (.03)	.141 (.04)	.109 (.05)	.126 (.06)
.0	.15	.166 (.04)	.171 (.05)	.146 (.06)	.193 (.07)
.0	.20	.164 (.03)	.169 (.04)	.198 (.05)	.222 (.06)
1.0	.00	.116 (.03)	.124 (.03)	.065 (.02)	.073 (.03)
1.0	.05	.117 (.03)	.119 (.03)	.075 (.04)	.082 (.04)
1.0	.10	.134 (.03)	.136 (.03)	.104 (.05)	.109 (.05)
1.0	.15	.157 (.04)	.167 (.05)	.140 (.06)	.169 (.06)
1.0	.20	.156 (.03)	.162 (.04)	.187 (.05)	.192 (.05)
2.0	.00	.118 (.03)	.126 (.03)	.059 (.02)	.081 (.04)
2.0	.05	.118 (.04)	.121 (.03)	.073 (.04)	.088 (.04)
2.0	.10	.136 (.03)	.142 (.03)	.101 (.04)	.108 (.05)
2.0	.15	.156 (.04)	.168 (.05)	.148 (.06)	.169 (.06)
2.0	.20	.156 (.03)	.161 (.03)	.171 (.04)	.196 (.06)

DIVERGENCE from the source PATTERN

δδ	W	OblmS/Eqmx	OblmS/Vmx	HyblP/Eqmx	HyblP/Vmx
.0	.00	18.0 (3.3)	18.7 (4.1)	10.1 (3.2)	13.6 (4.9)
.0	.05	18.1 (4.4)	18.4 (3.7)	12.4 (6.3)	13.3 (4.8)
.0	.10	20.3 (3.8)	21.1 (5.3)	15.0 (4.5)	17.2 (7.4)
.0	.15	25.6 (7.1)	26.5 (7.6)	20.9 (8.2)	26.2 (9.2)
.0	.20	25.8 (5.7)	26.4 (6.5)	27.5 (6.2)	29.3 (5.4)
1.0	.00	18.0 (3.6)	19.3 (4.8)	9.84 (2.5)	10.6 (3.3)
1.0	.05	18.2 (4.6)	18.5 (3.8)	11.4 (5.5)	11.8 (3.8)
1.0	.10	20.4 (3.8)	20.6 (4.2)	14.4 (5.0)	15.5 (6.2)
1.0	.15	24.6 (7.0)	25.9 (7.8)	20.1 (8.4)	23.7 (8.8)
1.0	.20	24.8 (5.2)	25.5 (5.9)	26.1 (4.7)	27.1 (6.6)
2.0	.00	18.4 (3.7)	19.6 (4.8)	9.16 (1.7)	11.1 (3.9)
2.0	.05	18.4 (4.6)	18.8 (3.8)	11.2 (5.0)	12.4 (4.5)
2.0	.10	20.6 (3.9)	21.4 (4.5)	14.1 (4.5)	15.2 (6.9)
2.0	.15	24.4 (7.1)	26.2 (7.5)	21.7 (9.0)	23.8 (8.1)
2.0	.20	24.8 (5.3)	25.5 (5.8)	25.2 (4.7)	28.7 (7.4)

MAXIMUM DIFFERENCE from the source PATTERN

δδ	W	OblmS/Eqmx	OblmS/Vmx	HyblP/Eqmx	HyblP/Vmx
.0	.00	.347 (.10)	.380 (.12)	.257 (.17)	.411 (.23)
.0	.05	.358 (.12)	.369 (.12)	.276 (.14)	.351 (.21)
.0	.10	.437 (.14)	.452 (.15)	.397 (.20)	.426 (.21)
.0	.15	.502 (.15)	.513 (.17)	.473 (.25)	.637 (.26)
.0	.20	.467 (.12)	.477 (.13)	.648 (.23)	.759 (.34)
1.0	.00	.336 (.10)	.373 (.13)	.224 (.14)	.245 (.16)
1.0	.05	.359 (.12)	.374 (.11)	.242 (.14)	.290 (.18)
1.0	.10	.430 (.14)	.441 (.14)	.376 (.20)	.371 (.17)
1.0	.15	.487 (.14)	.513 (.16)	.449 (.18)	.532 (.18)
1.0	.20	.459 (.12)	.464 (.12)	.607 (.21)	.631 (.20)
2.0	.00	.336 (.10)	.379 (.13)	.200 (.07)	.296 (.20)
2.0	.05	.364 (.12)	.383 (.11)	.241 (.12)	.327 (.22)
2.0	.10	.429 (.14)	.456 (.14)	.384 (.19)	.403 (.20)
2.0	.15	.485 (.14)	.517 (.15)	.457 (.19)	.535 (.19)
2.0	.20	.458 (.12)	.467 (.12)	.532 (.14)	.596 (.19)

RMS DIFFERENCE from the source COVARIANCES

δδ	W	OblmS/Eqmx	OblmS/Vmx	HyblP/Eqmx	HyblP/Vmx
.0	.00	.175 (.04)	.184 (.04)	.107 (.07)	.175 (.09)
.0	.05	.166 (.04)	.171 (.03)	.116 (.05)	.136 (.08)
.0	.10	.181 (.03)	.186 (.04)	.179 (.09)	.195 (.11)
.0	.15	.220 (.05)	.223 (.04)	.244 (.12)	.281 (.08)
.0	.20	.213 (.04)	.219 (.05)	.322 (.11)	.304 (.08)
1.0	.00	.171 (.03)	.181 (.04)	.095 (.04)	.106 (.06)
1.0	.05	.170 (.03)	.174 (.03)	.101 (.05)	.110 (.06)
1.0	.10	.184 (.03)	.185 (.03)	.166 (.10)	.166 (.09)
1.0	.15	.213 (.04)	.218 (.04)	.224 (.09)	.255 (.09)
1.0	.20	.211 (.04)	.214 (.05)	.293 (.07)	.297 (.08)
2.0	.00	.171 (.03)	.182 (.04)	.085 (.04)	.133 (.09)
2.0	.05	.172 (.03)	.176 (.03)	.099 (.05)	.128 (.08)
2.0	.10	.186 (.03)	.188 (.03)	.161 (.08)	.173 (.10)
2.0	.15	.212 (.04)	.223 (.04)	.232 (.07)	.260 (.10)
2.0	.20	.211 (.04)	.214 (.05)	.282 (.06)	.296 (.09)

TABLE 2

Dissimilarities among the Procrustes-start and 10 method-rated best Spin rotations ("ranks 0-10") of the same input pattern by the same rotation method, as well as, shown separately, between the first and last rotation ("Unord spin") in each Spin series. Means for each method at each hyperplane-noise level W, averaged over 10 repetitions of the 20 extraction patterns at this W, are given on three measures of pattern divergence, namely,

MIN DIVERG: Smallest congruence divergence between the solutions' matched factors.
AV DIVERG: Mean congruence divergence over the solutions' matched factors.
MAX DIVERG: Largest congruence divergence between the solutions' matched factors.

Orthomax variant was NORM-1 Equamax ($\gamma = 2.50$); Promax target was power 2 of Equamax solution
Oblimin variant was NORM-1 Quartimin ($\gamma = 0$)
Hyball variant was NORM-1 SCAN mode with <JA, JB, BH, CV, WSAL> = < 1, 2, .20, 1.0, 1.0>

W	Method	MIN DIVERG Ranks 0-10	MIN DIVERG Unord spin	AV DIVERG Ranks 0-10	AV DIVERG Unord spin	MAX DIVERG Ranks 0-10	MAX DIVERG Unord spin
.00	Equamax	.02	.02	.06	.06	.12	.11
	Promax	.01	.02	.06	.07	.12	.15
	Qmin-P	4.30	8.77	10.6	20.0	18.4	32.9
	Qmin-S	1.11	1.58	3.98	5.91	7.69	11.2
	Hybl-P	.44	1.92	1.43	13.2	3.25	31.2
	Hybl-S	.10	.77	.28	10.8	.73	27.6
.05	Equamax	.14	.13	.69	.62	1.48	1.32
	Promax	.10	.11	.72	.74	1.48	1.51
	Qmin-P	4.71	8.94	11.7	22.6	20.1	37.1
	Qmin-S	.93	1.33	3.53	5.42	6.69	10.6
	Hybl-P	.43	1.74	1.50	13.6	3.49	30.8
	Hybl-S	.10	1.57	.33	14.1	.83	33.0
.10	Equamax	.02	.02	.07	.07	.12	.12
	Promax	.01	.02	.06	.19	.11	.42
	Qmin-P	5.96	11.2	14.6	25.9	23.9	41.2
	Qmin-S	1.21	1.98	4.35	7.09	7.93	12.9
	Hybl-P	.82	4.11	4.65	21.7	11.8	43.4
	Hybl-S	.35	2.92	3.03	19.6	8.82	40.9
.15	Equamax	.08	.14	.19	.40	.33	.87
	Promax	.07	.08	.19	.22	.37	.42
	Qmin-P	7.91	14.6	17.5	28.6	29.3	43.9
	Qmin-S	1.97	2.69	5.81	8.89	11.0	16.3
	Hybl-P	2.56	8.47	12.3	27.9	27.5	48.4
	Hybl-S	1.82	7.02	9.93	26.5	23.0	48.1
.20	Equamax	.27	.23	1.02	.86	1.79	1.52
	Promax	.29	.11	1.02	.41	1.85	.75
	Qmin-P	6.88	11.7	16.3	25.7	28.0	41.0
	Qmin-S	1.78	3.14	5.84	10.6	10.7	19.9
	Hybl-P	2.85	6.96	16.6	26.1	35.4	47.5
	Hybl-S	2.17	6.55	14.9	26.9	33.4	47.9
All	Equamax	.11	.11	.41	.40	.77	.79
	Promax	.10	.07	.41	.32	.78	.65
	Qmin-P	5.95	11.0	14.2	24.6	23.9	39.2
	Qmin-S	1.40	2.15	4.70	7.59	8.84	14.2
	Hybl-P	1.42	4.64	7.30	20.5	16.3	40.3
	Hybl-S	.91	3.77	5.71	19.6	13.3	39.5

TABLE 3

Inaccuracy of source recovery by the Spin solutions of selected rotation methods, averaged over the extraction patterns from sources at each hyperplane-noise level W, and then averaged again over 10 repetitions of the latter. The standard error of each mean over its 10 repetitions, multiplied by 10, is given in parentheses.

Orthomax variant was NORM-1 Equamax ($\gamma = 2.50$); Promax target was power 2 of Equamax solution
Oblimin variant was NORM-1 Quartimin ($\gamma = .0$).
Hyball variant was NORM-1 SCAN mode with <JA,JB,BH,CV,WSAL> = < 1, 2, .20, 1.0, 1.0>.

"Rank 0" is rotation from Procrustes start; "Rank 1" is the Spin series' solution that optimized the method's criterion measure $\mathcal{L}$; "Best" is its solution closest to the source structure on the comparison at issue; "Rank of Best" shows how many Tries in the Spin series were rated superior by $\mathcal{L}$ to the actual best; and "Filtered Rnk" is the Best's rank in what remains of the Spin series when Tries are deleted if their Max Diverg from any retained Try of lower rank is less than 5.0°.

A. PATTERN, RMS DIFFERENCE

Mean (and 10*SE) RMS DIFFERENCE from the source PATTERN at each hyperplane noise level *W*.

W	Method	Rank 0	Rank 1	Best	Rank of Best	Filtered Rnk
.00	Equamax	.145 (.0)	.145 (.000)	.145 (.0)	21.4 (29.6)	1.99 (.300)
	Promax	.116 (.000)	.116 (.000)	.116 (.0)	20.2 (32.3)	1.97 (.335)
	Qmin-P	.092 (.000)	.135 (.025)	.103 (.011)	16.5 (23.7)	12.9 (14.2)
	Qmin-S	.112 (.000)	.132 (.015)	.110 (.003)	17.9 (16.6)	5.28 (3.52)
	Hybl-P	.053 (.0)	.055 (.001)	.054 (.001)	12.6 (13.2)	1.89 (.663)
	Hybl-S	.055 (.0)	.055 (.000)	.055 (.000)	11.4 (16.3)	1.87 (.600)
.05	Equamax	.143 (.0)	.143 (.001)	.142 (.005)	22.6 (36.6)	2.04 (.200)
	Promax	.116 (.0)	.117 (.002)	.116 (.007)	22.1 (47.4)	2.03 (.245)
	Qmin-P	.087 (.0)	.135 (.033)	.099 (.013)	17.3 (17.5)	16.0 (15.1)
	Qmin-S	.106 (.0)	.125 (.034)	.105 (.003)	25.2 (23.0)	5.29 (3.71)
	Hybl-P	.059 (.0)	.061 (.004)	.059 (.001)	6.67 (8.47)	1.99 (.986)
	Hybl-S	.064 (.000)	.063 (.003)	.063 (.003)	10.0 (18.1)	1.87 (.642)
.10	Equamax	.154 (.000)	.154 (.000)	.154 (.0)	21.3 (27.9)	1.94 (.490)
	Promax	.133 (.000)	.133 (.000)	.133 (.0)	20.4 (30.4)	1.95 (.548)
	Qmin-P	.100 (.000)	.157 (.035)	.114 (.014)	17.9 (14.8)	16.9 (12.3)
	Qmin-S	.124 (.0)	.143 (.020)	.121 (.005)	25.8 (18.5)	5.24 (3.09)
	Hybl-P	.070 (.000)	.082 (.031)	.072 (.008)	7.81 (9.38)	2.61 (2.72)
	Hybl-S	.074 (.0)	.084 (.033)	.075 (.013)	7.09 (9.73)	2.06 (1.49)
.15	Equamax	.173 (.000)	.173 (.002)	.173 (.001)	22.8 (19.7)	2.02 (.332)
	Promax	.154 (.0)	.154 (.003)	.153 (.001)	22.6 (16.6)	2.03 (.391)
	Qmin-P	.109 (.0)	.193 (.066)	.134 (.024)	20.0 (20.5)	19.6 (20.4)
	Qmin-S	.154 (.000)	.180 (.032)	.146 (.015)	25.4 (24.0)	8.80 (7.37)
	Hybl-P	.098 (.000)	.139 (.048)	.106 (.024)	6.93 (8.33)	4.55 (4.53)
	Hybl-S	.108 (.000)	.140 (.048)	.109 (.022)	8.71 (12.6)	4.03 (4.72)
.20	Equamax	.178 (.000)	.178 (.009)	.177 (.000)	23.4 (25.4)	2.13 (.510)
	Promax	.163 (.0)	.162 (.009)	.161 (.000)	22.5 (30.6)	2.13 (.808)
	Qmin-P	.110 (.0)	.177 (.040)	.133 (.013)	19.5 (20.4)	18.6 (18.4)
	Qmin-S	.147 (.0)	.166 (.025)	.143 (.018)	27.0 (24.5)	7.57 (8.19)
	Hybl-P	.131 (.0)	.183 (.092)	.133 (.034)	11.6 (17.8)	8.87 (11.4)
	Hybl-S	.146 (.0)	.187 (.067)	.136 (.026)	10.8 (17.2)	6.39 (10.5)
All	Equamax	.159 (.0)	.159 (.002)	.158 (.001)	22.3 (12.2)	2.02 (.211)
	Promax	.136 (.0)	.136 (.002)	.136 (.001)	21.6 (15.2)	2.02 (.225)
	Qmin-P	.100 (.000)	.159 (.020)	.116 (.008)	18.3 (9.93)	16.8 (8.76)
	Qmin-S	.128 (.000)	.149 (.008)	.125 (.005)	24.3 (10.8)	6.43 (3.05)
	Hybl-P	.083 (.000)	.104 (.020)	.085 (.009)	9.14 (5.03)	3.98 (1.73)
	Hybl-S	.089 (.000)	.106 (.014)	.087 (.007)	9.62 (6.75)	3.24 (2.38)

B. PATTERN, MAXIMUM DIFFERENCE

Mean (and 10*SE) MAXIMUM DIFFERENCE from the source PATTERN at each hyperplane noise level *W*.

W	Method	Rank 0	Rank 1	Best	Rank of Best	Filtered Rnk
.00	Equamax	.426 (.0)	.426 (.002)	.425 (.001)	6.41 (15.1)	1.62 (.844)
	Promax	.352 (.0)	.352 (.001)	.351 (.001)	4.91 (13.8)	1.45 (.650)
	Qmin-P	.270 (.000)	.429 (.120)	.288 (.043)	18.2 (26.5)	13.8 (17.7)
	Qmin-S	.337 (.0)	.439 (.088)	.323 (.024)	24.5 (22.6)	5.57 (5.51)
	Hybl-P	.179 (.0)	.185 (.013)	.172 (.010)	14.7 (20.1)	2.03 (.896)
	Hybl-S	.190 (.000)	.190 (.003)	.188 (.001)	7.73 (10.7)	1.80 (.922)
.05	Equamax	.433 (.0)	.435 (.011)	.431 (.023)	6.68 (25.1)	1.64 (1.05)
	Promax	.361 (.001)	.362 (.007)	.359 (.024)	7.25 (12.7)	1.57 (.748)
	Qmin-P	.254 (.0)	.422 (.176)	.276 (.044)	17.1 (13.5)	15.7 (10.7)
	Qmin-S	.314 (.000)	.379 (.102)	.303 (.026)	27.4 (19.5)	5.24 (3.06)
	Hybl-P	.195 (.000)	.199 (.019)	.187 (.006)	7.23 (7.21)	2.02 (1.14)
	Hybl-S	.213 (.0)	.212 (.016)	.207 (.014)	7.79 (15.4)	1.82 (1.20)
.10	Equamax	.487 (.0)	.487 (.002)	.486 (.000)	7.04 (13.1)	1.64 (.768)
	Promax	.448 (.000)	.448 (.002)	.447 (.000)	5.67 (18.5)	1.46 (.768)
	Qmin-P	.307 (.0)	.499 (.068)	.323 (.060)	19.3 (15.4)	18.2 (13.7)
	Qmin-S	.400 (.001)	.459 (.065)	.377 (.034)	24.6 (18.3)	5.17 (3.51)
	Hybl-P	.232 (.000)	.285 (.100)	.228 (.057)	8.10 (7.85)	2.84 (3.94)
	Hybl-S	.249 (.0)	.296 (.148)	.243 (.061)	7.28 (9.98)	2.24 (3.08)
.15	Equamax	.529 (.0)	.530 (.019)	.527 (.002)	6.38 (12.9)	1.66 (1.09)
	Promax	.476 (.001)	.478 (.023)	.474 (.002)	8.76 (20.1)	1.81 (.831)
	Qmin-P	.337 (.0)	.561 (.261)	.378 (.077)	19.6 (14.9)	19.2 (14.8)
	Qmin-S	.457 (.001)	.537 (.107)	.417 (.073)	27.3 (18.9)	8.95 (6.15)
	Hybl-P	.304 (.000)	.458 (.131)	.330 (.097)	9.44 (12.3)	6.16 (7.70)
	Hybl-S	.341 (.0)	.453 (.157)	.336 (.099)	8.93 (17.1)	4.35 (6.80)
.20	Equamax	.548 (.001)	.542 (.055)	.532 (.004)	7.06 (15.8)	1.77 (.955)
	Promax	.503 (.0)	.493 (.064)	.483 (.000)	6.55 (10.0)	1.67 (.900)
	Qmin-P	.325 (.001)	.509 (.159)	.363 (.056)	19.9 (18.9)	19.0 (20.1)
	Qmin-S	.428 (.000)	.474 (.080)	.398 (.035)	24.9 (26.7)	7.55 (7.31)
	Hybl-P	.427 (.0)	.586 (.373)	.394 (.139)	13.1 (20.2)	9.89 (13.4)
	Hybl-S	.461 (.0)	.598 (.173)	.406 (.119)	13.2 (21.1)	7.51 (10.4)
All	Equamax	.484 (.001)	.484 (.009)	.480 (.005)	6.71 (8.08)	1.66 (.478)
	Promax	.428 (.0)	.427 (.012)	.423 (.005)	6.63 (6.86)	1.59 (.393)
	Qmin-P	.299 (.0)	.484 (.087)	.325 (.034)	18.8 (9.07)	17.2 (8.62)
	Qmin-S	.387 (.0)	.457 (.039)	.364 (.020)	25.7 (8.50)	6.50 (2.47)
	Hybl-P	.267 (.000)	.343 (.084)	.262 (.044)	10.5 (7.27)	4.59 (2.81)
	Hybl-S	.291 (.0)	.350 (.048)	.276 (.036)	8.99 (6.99)	3.54 (3.43)

C. PATTERN, CONGRUENCE DIVERGENCE

Mean (and 10*SE) CONGRUENCE DIVERGENCE from the source PATTERN at each hyperplane noise level *W*.

W	Method	Rank 0	Rank 1	Best	Rank of Best	Filtered Rnk
.00	Equamax	22.7 (.026)	22.7 (.032)	22.7 (.026)	20.9 (35.5)	1.97 (.332)
	Promax	17.6 (.038)	17.6 (.034)	17.6 (.006)	20.0 (40.3)	1.98 (.332)
	Qmin-P	14.3 (.0)	20.6 (3.57)	15.8 (2.10)	15.3 (18.5)	12.1 (12.2)
	Qmin-S	17.3 (.010)	20.1 (2.02)	17.0 (.653)	16.6 (16.6)	5.22 (4.19)
	Hybl-P	8.50 (.019)	8.81 (.213)	8.63 (.160)	12.7 (16.9)	1.94 (.735)
	Hybl-S	8.78 (.019)	8.78 (.034)	8.75 (.029)	12.7 (22.2)	1.86 (.709)
.05	Equamax	22.4 (.0)	22.4 (.105)	22.3 (.667)	21.0 (34.4)	2.02 (.250)
	Promax	17.8 (.024)	17.9 (.601)	17.8 (.741)	20.8 (40.4)	2.02 (.335)
	Qmin-P	13.6 (.0)	20.8 (5.20)	15.3 (1.71)	16.6 (17.0)	15.3 (15.0)
	Qmin-S	16.5 (.024)	19.5 (5.73)	16.2 (.400)	24.1 (23.4)	5.24 (3.64)
	Hybl-P	9.37 (.004)	9.61 (.471)	9.37 (.164)	6.80 (11.4)	1.99 (.970)
	Hybl-S	9.93 (.012)	9.88 (.477)	9.80 (.448)	11.0 (13.7)	1.89 (.850)
.10	Equamax	23.4 (.028)	23.4 (.051)	23.4 (.020)	21.3 (31.0)	1.94 (.436)
	Promax	19.8 (.0)	19.8 (.067)	19.7 (.0)	21.2 (23.1)	1.96 (.391)
	Qmin-P	15.2 (.025)	23.9 (6.14)	17.3 (2.09)	17.6 (18.0)	16.6 (16.1)
	Qmin-S	18.8 (.029)	21.7 (3.82)	18.5 (.829)	24.9 (20.0)	5.04 (2.28)
	Hybl-P	10.6 (.0)	12.2 (5.16)	10.8 (1.13)	7.74 (8.89)	2.73 (2.95)
	Hybl-S	11.2 (.0)	12.5 (3.68)	11.2 (1.33)	7.77 (9.22)	2.34 (2.17)
.15	Equamax	26.6 (.014)	26.7 (.271)	26.6 (.104)	22.3 (29.9)	2.02 (.335)
	Promax	23.9 (.055)	23.9 (.481)	23.8 (.116)	22.1 (29.7)	2.04 (.300)
	Qmin-P	16.5 (.0)	29.4 (10.1)	20.5 (3.63)	19.5 (15.6)	19.1 (15.9)
	Qmin-S	23.7 (.0)	27.6 (4.89)	22.5 (1.84)	24.3 (23.9)	8.59 (6.04)
	Hybl-P	14.6 (.007)	19.8 (6.73)	15.6 (3.28)	7.35 (10.0)	4.79 (7.21)
	Hybl-S	15.8 (.010)	19.8 (7.41)	16.0 (3.06)	8.38 (13.0)	3.90 (5.41)
.20	Equamax	27.6 (.050)	27.5 (1.62)	27.3 (.043)	22.2 (28.2)	2.11 (.539)
	Promax	25.0 (.0)	24.9 (1.66)	24.7 (.0)	21.6 (36.3)	2.13 (.808)
	Qmin-P	17.2 (.0)	27.7 (6.09)	20.6 (1.92)	18.8 (17.6)	17.9 (17.0)
	Qmin-S	23.1 (.0)	25.9 (3.46)	22.4 (2.15)	28.0 (23.4)	7.97 (9.63)
	Hybl-P	19.2 (.0)	26.9 (12.6)	19.8 (3.95)	12.5 (14.9)	9.54 (8.71)
	Hybl-S	20.8 (.0)	27.3 (9.43)	20.0 (2.82)	11.7 (20.3)	6.96 (11.4)
All	Equamax	24.6 (.021)	24.6 (.316)	24.5 (.146)	21.6 (17.2)	2.01 (.206)
	Promax	20.8 (.027)	20.8 (.315)	20.7 (.158)	21.1 (15.5)	2.02 (.217)
	Qmin-P	15.4 (.0)	24.5 (2.92)	17.9 (1.28)	17.6 (9.90)	16.2 (8.87)
	Qmin-S	19.9 (.0)	23.0 (1.30)	19.3 (.645)	23.6 (8.08)	6.41 (1.93)
	Hybl-P	12.4 (.015)	15.5 (2.62)	12.8 (1.24)	9.44 (5.37)	4.19 (1.52)
	Hybl-S	13.3 (.0)	15.6 (2.30)	13.1 (.858)	10.3 (6.21)	3.39 (2.29)

D. COVARIANCES, RMS DIFFERENCE

Mean (and 10*SE) RMS DIFFERENCE from the source COVARIANCES at each hyperplane noise level *W*.

W	Method	Rank 0	Rank 1	Best	Rank of Best	Filtered Rnk
.00	Equamax	.296 (.001)	.296 (.000)	.296 (.0)	20.4 (17.8)	1.97 (.332)
	Promax	.182 (.0)	.182 (.0)	.182 (.0)	20.7 (36.3)	1.98 (.332)
	Qmin-P	.162 (.000)	.199 (.033)	.152 (.011)	29.9 (23.4)	22.7 (17.8)
	Qmin-S	.175 (.000)	.196 (.027)	.168 (.014)	30.7 (22.9)	6.60 (4.84)
	Hybl-P	.073 (.000)	.078 (.006)	.070 (.007)	17.0 (20.9)	2.35 (3.59)
	Hybl-S	.079 (.0)	.079 (.002)	.078 (.001)	15.4 (23.2)	1.90 (.850)
.05	Equamax	.277 (.0)	.277 (.000)	.277 (.000)	21.6 (26.8)	2.01 (.374)
	Promax	.167 (.0)	.169 (.011)	.167 (.000)	21.7 (7.49)	2.02 (.335)
	Qmin-P	.149 (.0)	.185 (.042)	.144 (.014)	24.3 (18.6)	22.5 (17.5)
	Qmin-S	.161 (.0)	.178 (.042)	.154 (.009)	29.2 (38.2)	5.40 (5.09)
	Hybl-P	.075 (.000)	.078 (.016)	.071 (.007)	9.85 (13.8)	2.19 (1.21)
	Hybl-S	.087 (.000)	.086 (.010)	.083 (.010)	14.4 (17.8)	1.98 (.976)
.10	Equamax	.271 (.0)	.271 (.0)	.271 (.0)	21.7 (37.0)	1.99 (.200)
	Promax	.186 (.0)	.186 (.001)	.185 (.001)	21.6 (33.0)	1.96 (.450)
	Qmin-P	.163 (.0)	.192 (.039)	.153 (.016)	22.6 (20.2)	21.5 (19.1)
	Qmin-S	.174 (.0)	.188 (.025)	.165 (.015)	23.2 (20.4)	5.21 (4.02)
	Hybl-P	.105 (.0)	.130 (.041)	.103 (.024)	9.63 (8.46)	3.57 (5.78)
	Hybl-S	.121 (.000)	.137 (.037)	.113 (.018)	10.5 (15.3)	3.41 (4.45)
.15	Equamax	.283 (.0)	.283 (.000)	.283 (.0)	20.4 (28.9)	1.99 (.610)
	Promax	.211 (.000)	.211 (.001)	.211 (.000)	22.1 (33.0)	2.00 (.350)
	Qmin-P	.180 (.0)	.235 (.038)	.165 (.024)	26.8 (31.0)	26.4 (30.3)
	Qmin-S	.207 (.000)	.229 (.064)	.188 (.022)	26.2 (32.8)	8.47 (12.0)
	Hybl-P	.170 (.0)	.229 (.076)	.155 (.038)	13.2 (21.5)	8.20 (15.9)
	Hybl-S	.186 (.000)	.227 (.074)	.161 (.036)	12.4 (17.5)	5.29 (6.97)
.20	Equamax	.288 (.0)	.288 (.0)	.288 (.000)	22.6 (29.7)	2.11 (.450)
	Promax	.218 (.0)	.216 (.023)	.212 (.0)	23.4 (20.6)	2.17 (.400)
	Qmin-P	.190 (.0)	.225 (.068)	.172 (.034)	29.1 (19.2)	27.6 (17.2)
	Qmin-S	.212 (.000)	.216 (.036)	.192 (.011)	24.2 (19.3)	7.62 (4.25)
	Hybl-P	.221 (.0)	.280 (.133)	.182 (.042)	17.9 (22.9)	13.3 (20.6)
	Hybl-S	.226 (.000)	.290 (.126)	.185 (.051)	18.0 (18.1)	9.87 (9.77)
All	Equamax	.283 (.000)	.283 (.000)	.283 (.0)	21.3 (12.6)	2.01 (.174)
	Promax	.193 (.0)	.193 (.005)	.191 (.000)	21.9 (11.9)	2.02 (.187)
	Qmin-P	.169 (.0)	.207 (.024)	.157 (.007)	26.6 (10.8)	24.1 (9.36)
	Qmin-S	.186 (.000)	.201 (.017)	.173 (.008)	26.7 (15.5)	6.66 (3.42)
	Hybl-P	.129 (.000)	.159 (.035)	.116 (.010)	13.5 (9.22)	5.92 (5.37)
	Hybl-S	.140 (.0)	.164 (.026)	.124 (.017)	14.1 (7.53)	4.49 (1.73)

E. COVARIANCES, MAXIMUM DIFFERENCE

Mean (and 10*SE) MAXIMUM DIFFERENCE from the source COVARIANCES at each hyperplane noise level *W*.

W	Method	Rank 0	Rank 1	Best	Rank of Best	Filtered Rnk
.00	Equamax	.544 (.001)	.544 (.001)	.544 (.0)	20.4 (18.6)	1.96 (.374)
	Promax	.347 (.0)	.347 (.000)	.347 (.000)	19.8 (34.7)	1.97 (.332)
	Qmin-P	.303 (.0)	.382 (.074)	.269 (.040)	30.3 (34.3)	23.4 (23.0)
	Qmin-S	.326 (.0)	.379 (.095)	.308 (.028)	29.0 (18.2)	6.42 (4.20)
	Hybl-P	.142 (.000)	.150 (.023)	.129 (.019)	18.1 (11.9)	2.47 (3.18)
	Hybl-S	.152 (.0)	.153 (.004)	.148 (.004)	16.8 (22.6)	1.92 (.600)
.05	Equamax	.518 (.001)	.518 (.001)	.518 (.0)	21.9 (17.8)	2.01 (.320)
	Promax	.294 (.000)	.296 (.010)	.294 (.000)	23.3 (32.1)	2.04 (.200)
	Qmin-P	.269 (.000)	.338 (.085)	.241 (.029)	25.5 (20.6)	23.5 (18.4)
	Qmin-S	.296 (.001)	.327 (.086)	.268 (.030)	31.4 (34.8)	6.03 (7.11)
	Hybl-P	.148 (.000)	.149 (.051)	.125 (.009)	9.91 (10.9)	2.23 (1.14)
	Hybl-S	.168 (.000)	.169 (.029)	.159 (.029)	13.7 (20.0)	2.04 (1.03)
.10	Equamax	.515 (.001)	.515 (.001)	.515 (.0)	21.4 (17.9)	1.98 (.245)
	Promax	.353 (.000)	.353 (.001)	.352 (.000)	20.7 (32.3)	1.96 (.450)
	Qmin-P	.300 (.001)	.373 (.090)	.259 (.035)	25.2 (21.7)	23.7 (19.1)
	Qmin-S	.326 (.000)	.360 (.064)	.291 (.030)	27.5 (24.9)	6.02 (3.90)
	Hybl-P	.197 (.000)	.253 (.079)	.191 (.050)	8.62 (12.5)	3.26 (3.80)
	Hybl-S	.238 (.000)	.269 (.091)	.217 (.025)	12.4 (20.8)	3.69 (7.82)
.15	Equamax	.501 (.001)	.501 (.001)	.501 (.001)	20.6 (32.8)	2.01 (.436)
	Promax	.370 (.0)	.370 (.002)	.369 (.003)	22.7 (35.0)	2.01 (.300)
	Qmin-P	.331 (.0)	.439 (.124)	.285 (.049)	26.6 (19.7)	26.2 (19.3)
	Qmin-S	.373 (.0)	.430 (.101)	.329 (.047)	32.7 (25.5)	9.97 (9.38)
	Hybl-P	.335 (.0)	.439 (.139)	.283 (.092)	13.1 (18.9)	8.46 (14.5)
	Hybl-S	.358 (.0)	.437 (.124)	.295 (.083)	11.8 (23.3)	5.54 (11.0)
.20	Equamax	.550 (.0)	.550 (.000)	.550 (.0)	22.6 (41.0)	2.11 (.700)
	Promax	.410 (.001)	.402 (.064)	.391 (.003)	22.5 (26.0)	2.16 (.624)
	Qmin-P	.349 (.000)	.426 (.138)	.293 (.063)	29.1 (23.5)	27.4 (23.7)
	Qmin-S	.395 (.001)	.410 (.096)	.340 (.038)	29.7 (32.9)	9.47 (8.63)
	Hybl-P	.426 (.000)	.546 (.320)	.329 (.099)	17.5 (21.6)	12.5 (18.2)
	Hybl-S	.420 (.000)	.557 (.270)	.335 (.091)	18.3 (16.9)	9.96 (8.54)
All	Equamax	.526 (.001)	.526 (.001)	.526 (.001)	21.4 (15.6)	2.01 (.233)
	Promax	.355 (.0)	.354 (.013)	.350 (.001)	21.8 (10.8)	2.02 (.170)
	Qmin-P	.311 (.000)	.392 (.046)	.269 (.021)	27.3 (10.4)	24.8 (8.72)
	Qmin-S	.343 (.0)	.381 (.039)	.307 (.016)	30.1 (16.8)	7.58 (3.28)
	Hybl-P	.250 (.000)	.308 (.084)	.211 (.026)	13.4 (7.61)	5.79 (4.97)
	Hybl-S	.268 (.0)	.317 (.055)	.231 (.031)	14.6 (7.85)	4.63 (3.20)

TABLE 4

Differences among the distinguished Spin solutions whose similarities to source are reported in Table 3.

Mean (and 10*SE) Min/Av/Max Divergence in each Spin set between the following special solutions:
- Sors: The source pattern.
- Rnk1: Rank 1 on method's quality measure.
- DivP: Best on pattern Divergence from source.
- RmsC: Best on RMS covariance difference from source.

A. SMALLEST PATTERN-COLUMN DIVERGENCE

W	Method	Sors-Rnk1	Sors-DivP	Sors-RmsC	Rnk1-DivP	Rnk1-RmsC	DivP-RmsC
.05	Qmin-P	11.2 (3.18)	9.62 (2.54)	10.8 (6.30)	5.57 (6.92)	5.89 (3.70)	5.07 (4.87)
	Qmin-S	10.9 (2.57)	10.1 (1.14)	10.2 (1.12)	1.36 (2.56)	1.89 (3.58)	1.40 (3.55)
	Hybl-P	6.46 (.433)	6.40 (.376)	6.47 (.444)	.356 (.680)	.402 (.644)	.380 (.411)
	Hybl-S	6.55 (.090)	6.54 (.102)	6.56 (.130)	.084 (.078)	.106 (.291)	.082 (.281)
.10	Qmin-P	14.2 (6.30)	10.5 (3.22)	12.9 (7.09)	6.21 (4.60)	8.07 (7.96)	6.24 (13.6)
	Qmin-S	13.0 (5.16)	10.7 (1.96)	11.5 (2.09)	2.22 (3.28)	2.42 (4.43)	1.80 (4.17)
	Hybl-P	7.38 (1.41)	7.12 (1.27)	7.40 (2.50)	.651 (1.29)	.717 (1.13)	.537 (1.18)
	Hybl-S	7.42 (.958)	7.16 (.398)	7.67 (3.22)	.344 (.568)	.449 (1.68)	.368 (1.61)
.15	Qmin-P	16.9 (6.48)	12.5 (6.11)	17.7 (16.2)	10.2 (9.94)	12.7 (16.6)	10.2 (21.3)
	Qmin-S	15.6 (4.44)	13.5 (2.69)	14.9 (4.46)	3.94 (6.38)	4.52 (9.43)	3.09 (4.38)
	Hybl-P	9.85 (3.13)	9.46 (1.81)	10.6 (5.79)	1.66 (5.28)	3.07 (10.4)	2.50 (9.11)
	Hybl-S	9.80 (4.23)	9.50 (1.83)	10.3 (5.49)	1.17 (4.15)	1.72 (8.23)	1.37 (6.28)
.20	Qmin-P	17.3 (6.91)	13.0 (4.53)	17.9 (9.03)	7.14 (10.0)	11.2 (17.0)	10.3 (17.0)
	Qmin-S	16.4 (3.58)	14.4 (3.32)	15.7 (2.55)	2.81 (4.47)	3.31 (4.16)	3.17 (4.01)
	Hybl-P	15.7 (9.99)	12.0 (3.62)	14.1 (6.76)	2.82 (10.7)	4.45 (14.8)	3.43 (12.9)
	Hybl-S	14.7 (4.71)	12.1 (1.82)	13.7 (6.01)	1.62 (4.68)	2.84 (7.91)	2.37 (7.49)

B. MEAN PATTERN-COLUMN DIVERGENCE

W	Method	Sors-Rnk1	Sors-DivP	Sors-RmsC	Rnk1-DivP	Rnk1-RmsC	DivP-RmsC
.05	Qmin-P	20.8 (5.20)	15.3 (1.71)	19.2 (7.02)	13.7 (8.51)	17.8 (7.75)	12.2 (14.8)
	Qmin-S	19.5 (5.73)	16.2 (.400)	18.6 (5.43)	6.43 (8.79)	10.7 (15.5)	6.69 (13.1)
	Hybl-P	9.61 (.471)	9.37 (.164)	9.54 (.168)	1.27 (1.49)	1.50 (1.67)	1.18 (1.09)
	Hybl-S	9.88 (.477)	9.80 (.448)	10.0 (3.86)	.401 (.266)	.750 (5.04)	.510 (4.95)
.10	Qmin-P	23.9 (6.14)	17.3 (2.09)	23.6 (11.3)	16.0 (8.49)	18.7 (17.8)	16.2 (21.8)
	Qmin-S	21.7 (3.82)	18.5 (.829)	20.7 (7.94)	8.99 (10.8)	9.69 (19.8)	6.88 (18.2)
	Hybl-P	12.2 (5.16)	10.8 (1.13)	11.8 (5.40)	3.59 (7.14)	4.35 (7.33)	2.83 (6.34)
	Hybl-S	12.5 (3.68)	11.2 (1.33)	12.9 (7.41)	2.94 (5.19)	4.07 (12.2)	3.18 (11.6)
.15	Qmin-P	29.4 (10.1)	20.5 (3.63)	29.1 (11.4)	21.9 (16.3)	27.0 (14.0)	22.1 (25.8)
	Qmin-S	27.6 (4.89)	22.5 (1.84)	25.8 (6.44)	12.8 (16.1)	14.8 (21.3)	11.6 (12.7)
	Hybl-P	19.8 (6.73)	15.6 (3.28)	20.5 (13.1)	9.87 (9.99)	14.8 (16.8)	11.6 (17.8)
	Hybl-S	19.8 (7.41)	16.0 (3.06)	19.4 (5.46)	8.99 (14.6)	11.5 (16.7)	8.09 (13.8)
.20	Qmin-P	27.7 (6.09)	20.6 (1.92)	29.7 (14.6)	17.6 (11.8)	25.0 (23.5)	23.4 (23.5)
	Qmin-S	25.9 (3.46)	22.4 (2.15)	25.7 (4.07)	10.9 (14.1)	10.7 (9.44)	10.2 (10.6)
	Hybl-P	26.9 (12.6)	19.8 (3.95)	25.5 (14.0)	16.9 (21.8)	20.9 (30.5)	15.7 (21.2)
	Hybl-S	27.3 (9.43)	20.0 (2.82)	25.7 (7.62)	16.8 (18.8)	20.5 (18.0)	14.9 (12.8)

C. LARGEST PATTERN-COLUMN DIVERGENCE

W	Method	Sors–Rnk1	Sors–DivP	Sors–RmsC	Rnk1–DivP	Rnk1–RmsC	DivP–RmsC
.00	Qmin-P	31.7 (8.32)	23.4 (5.35)	31.5 (19.8)	21.3 (12.8)	30.2 (14.9)	23.0 (18.6)
	Qmin-S	31.6 (4.40)	25.6 (2.34)	28.6 (8.44)	14.1 (10.4)	17.8 (17.6)	9.74 (24.9)
	Hybl-P	11.5 (.526)	11.3 (.506)	12.0 (4.72)	2.38 (2.91)	3.53 (10.5)	3.18 (9.02)
	Hybl-S	11.5 (.127)	11.5 (.078)	11.5 (.128)	.358 (.378)	.422 (.502)	.355 (.625)
.05	Qmin-P	32.4 (7.56)	22.7 (4.45)	29.1 (12.8)	23.8 (16.3)	31.5 (16.0)	20.0 (26.9)
	Qmin-S	29.9 (10.9)	24.1 (1.69)	29.2 (10.8)	12.6 (19.2)	21.8 (30.4)	13.4 (25.6)
	Hybl-P	13.6 (1.37)	13.1 (.483)	13.5 (.735)	2.95 (4.07)	3.59 (3.58)	2.61 (2.61)
	Hybl-S	14.6 (1.38)	14.4 (1.42)	14.9 (7.83)	1.38 (.925)	2.25 (12.7)	1.23 (12.1)
.10	Qmin-P	36.3 (10.3)	25.1 (6.71)	36.3 (17.5)	26.4 (19.8)	30.2 (27.4)	26.7 (32.4)
	Qmin-S	32.8 (5.61)	26.9 (2.46)	30.8 (16.6)	16.3 (19.5)	17.9 (40.2)	12.9 (35.0)
	Hybl-P	18.5 (7.71)	15.4 (4.35)	17.2 (11.3)	8.56 (13.3)	10.4 (15.6)	6.86 (14.4)
	Hybl-S	19.1 (6.32)	16.2 (5.08)	19.0 (11.9)	8.24 (11.6)	9.98 (21.1)	7.17 (21.7)
.15	Qmin-P	43.9 (18.8)	29.1 (7.08)	42.3 (17.1)	36.0 (29.2)	42.3 (18.8)	34.8 (30.5)
	Qmin-S	42.0 (9.69)	32.3 (6.40)	38.1 (11.7)	24.3 (35.2)	27.9 (39.0)	21.6 (20.9)
	Hybl-P	33.7 (12.4)	23.9 (7.61)	33.2 (22.4)	23.2 (20.3)	32.7 (26.0)	24.2 (27.9)
	Hybl-S	33.8 (12.7)	24.1 (8.09)	30.8 (14.5)	23.4 (27.1)	28.4 (22.0)	18.3 (21.1)
.20	Qmin-P	39.5 (11.2)	29.2 (5.21)	43.3 (24.7)	30.2 (19.4)	39.4 (30.6)	37.2 (42.5)
	Qmin-S	35.4 (7.12)	30.6 (3.06)	35.9 (6.17)	20.2 (28.8)	19.4 (19.8)	18.9 (22.5)
	Hybl-P	42.1 (21.4)	30.3 (17.3)	39.0 (20.0)	36.9 (32.0)	41.2 (34.6)	31.1 (23.8)
	Hybl-S	42.9 (16.1)	30.8 (11.3)	39.6 (15.8)	37.6 (25.5)	42.5 (30.9)	30.4 (18.6)

D. RMS FACTOR-COVARIANCE DIFFERENCE

W	Method	Sors–Rnk1	Sors–DivP	Sors–RmsC	Rnk1–DivP	Rnk1–RmsC	DivP–RmsC
.05	Qmin-P	.185 (.042)	.157 (.019)	.144 (.014)	.071 (.051)	.091 (.043)	.052 (.053)
	Qmin-S	.178 (.042)	.159 (.008)	.154 (.009)	.039 (.063)	.062 (.097)	.032 (.060)
	Hybl-P	.078 (.016)	.075 (.011)	.071 (.007)	.019 (.028)	.022 (.025)	.016 (.016)
	Hybl-S	.086 (.010)	.084 (.011)	.083 (.010)	.010 (.005)	.014 (.057)	.007 (.055)
.10	Qmin-P	.192 (.039)	.171 (.020)	.153 (.016)	.068 (.070)	.081 (.079)	.065 (.080)
	Qmin-S	.188 (.025)	.173 (.006)	.165 (.015)	.043 (.053)	.046 (.094)	.030 (.072)
	Hybl-P	.130 (.041)	.114 (.032)	.103 (.024)	.058 (.094)	.067 (.090)	.039 (.059)
	Hybl-S	.137 (.037)	.122 (.038)	.113 (.018)	.051 (.071)	.057 (.089)	.037 (.101)
.15	Qmin-P	.235 (.038)	.195 (.046)	.165 (.024)	.114 (.084)	.134 (.109)	.102 (.087)
	Qmin-S	.229 (.064)	.206 (.039)	.188 (.022)	.076 (.131)	.089 (.164)	.064 (.069)
	Hybl-P	.229 (.076)	.186 (.064)	.155 (.038)	.149 (.166)	.192 (.198)	.134 (.154)
	Hybl-S	.227 (.074)	.183 (.088)	.161 (.036)	.148 (.202)	.172 (.138)	.111 (.155)
.20	Qmin-P	.225 (.068)	.205 (.022)	.172 (.034)	.097 (.083)	.127 (.130)	.114 (.153)
	Qmin-S	.216 (.036)	.208 (.019)	.192 (.011)	.060 (.092)	.061 (.065)	.057 (.069)
	Hybl-P	.280 (.133)	.227 (.040)	.182 (.042)	.236 (.216)	.247 (.235)	.179 (.179)
	Hybl-S	.290 (.126)	.231 (.070)	.185 (.051)	.246 (.194)	.265 (.223)	.175 (.136)

TABLE 5

Marginal means, SDs, and correlations among the five inaccuracy measures for each method variant, computed over all the method's Spin solutions, or all the selected solutions, at all W-levels.

All Spin solutions (unfiltered)

Equamax	Mean	SD	Correlations				
RmsP	.16	.04	1.00				
MaxP	.49	.14	.86	1.00			
DivP	24.8	5.4	.96	.76	1.00		
RmsC	.28	.04	.18	.15	.20	1.00	
MaxC	.53	.09	.04	.08	.05	.68	1.00

Promax	Mean	SD	Correlations				
RmsP	.14	.04	1.00				
MaxP	.43	.15	.91	1.00			
DivP	21.1	6.4	.97	.82	1.00		
RmsC	.19	.04	.64	.62	.64	1.00	
MaxC	.36	.09	.56	.60	.53	.83	1.00

Qmin-P	Mean	SD	Correlations				
RmsP	.17	.06	1.00				
MaxP	.53	.18	.89	1.00			
DivP	27.4	8.5	.98	.82	1.00		
RmsC	.21	.05	.60	.55	.60	1.00	
MaxC	.40	.11	.54	.51	.53	.87	1.00

Qmin-S	Mean	SD	Correlations				
RmsP	.15	.04	1.00				
MaxP	.44	.15	.88	1.00			
DivP	22.6	6.6	.97	.80	1.00		
RmsC	.20	.05	.63	.53	.62	1.00	
MaxC	.36	.10	.56	.50	.54	.86	1.00

Hybl-P	Mean	SD	Correlations				
RmsP	.17	.09	1.00				
MaxP	.56	.29	.93	1.00			
DivP	22.6	11.0	.96	.84	1.00		
RmsC	.25	.13	.88	.86	.83	1.00	
MaxC	.47	.25	.85	.85	.80	.95	1.00

Hybl-S	Mean	SD	Correlations				
RmsP	.17	.09	1.00				
MaxP	.56	.31	.93	1.00			
DivP	21.8	11.0	.93	.81	1.00		
RmsC	.25	.14	.90	.88	.82	1.00	
MaxC	.48	.27	.86	.86	.79	.96	1.00

First 10 in Spin Cream filtered at 5.0°

Equamax	Mean	SD	Correlations				
RmsP	.16	.04	1.00				
MaxP	.50	.14	.86	1.00			
DivP	25.3	5.7	.96	.78	1.00		
RmsC	.29	.04	.26	.21	.27	1.00	
MaxC	.53	.09	.08	.08	.08	.66	1.00

Promax	Mean	SD	Correlations				
RmsP	.14	.04	1.00				
MaxP	.44	.16	.91	1.00			
DivP	21.7	6.8	.97	.83	1.00		
RmsC	.20	.05	.67	.66	.68	1.00	
MaxC	.36	.09	.58	.62	.55	.83	1.00

Qmin-P	Mean	SD	Correlations				
RmsP	.16	.05	1.00				
MaxP	.48	.16	.88	1.00			
DivP	24.3	7.0	.97	.80	1.00		
RmsC	.21	.06	.59	.53	.58	1.00	
MaxC	.39	.11	.54	.49	.53	.86	1.00

Qmin-S	Mean	SD	Correlations				
RmsP	.17	.04	1.00				
MaxP	.51	.16	.86	1.00			
DivP	25.4	6.7	.97	.77	1.00		
RmsC	.21	.05	.56	.50	.55	1.00	
MaxC	.39	.10	.52	.51	.51	.87	1.00

Hybl-P	Mean	SD	Correlations				
RmsP	.16	.06	1.00				
MaxP	.58	.23	.87	1.00			
DivP	21.9	8.0	.92	.69	1.00		
RmsC	.26	.10	.80	.77	.71	1.00	
MaxC	.50	.21	.74	.75	.65	.93	1.00

Hybl-S	Mean	SD	Correlations				
RmsP	.19	.06	1.00				
MaxP	.65	.24	.85	1.00			
DivP	23.9	7.8	.85	.56	1.00		
RmsC	.29	.10	.76	.75	.59	1.00	
MaxC	.56	.21	.70	.71	.52	.92	1.00

TABLE 6

Percentage distributions of unsigned loadings in the source patterns both in population and in sample (NS = 400), broken down by hyperplane-noise level crossed with the Salient/Nonsaliant difference in production parameters.

Strip PN: Percent source-pattern loadings in the 20 populations at this W-level generated as Nonsalient.
Strip PS: Percent source-pattern loadings in the 20 populations at this W-level generated as Salient.
Strip SN: Percent procrustes-pattern loadings in the 20 size-400 samples at this W-level generated as Nonsalient.
Strip SS: Percent procrustes-pattern loadings in the 20 size-400 samples at this W-level generated as Salient.

	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35	37	39	41	43	45	47	49	51	53	55	57	59	61	63	65	67	69	71	73	75	77	79	> 80
PN	56																																								
PS										1	1	1	1	1	2	2	2	2	2	2	2	2	2	2	2	1	1	2	1	1	1	1	1	1	1	1	1	1	1	1	4
SN	19	15	10	6	3	2																																			
SS										1	1	1	1	1	2	2	2	1	2	2	2	2	1	2	2	1	2	1	1	1	1	1	1	1	1	1	1	1	1	1	3

Hyperplane-noise level .00. Scatter of loading magnitudes in intervals .01(j ± 1) for j = 1,3,5,... .

	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35	37	39	41	43	45	47	49	51	53	55	57	59	61	63	65	67	69	71	73	75	77	79	> 80
PN	29	19	7	1																																					
PS									1	1	1	1	1	1	1	2	2	2	1	1	2	2	2	2	2	2	1	1	2	1	1	1	1	1	1	1	1	1	1	1	3
SN	15	14	10	7	4	2	1	1																																	
SS										1	1	1	1	1	1	1	2	1	1	2	2	2	2	2	1	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	3

Hyperplane-noise level .05. Scatter of loading magnitudes in intervals .01(j ± 1) for j = 1,3,5,... .

	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35	37	39	41	43	45	47	49	51	53	55	57	59	61	63	65	67	69	71	73	75	77	79	> 80
PN	12	14	12	9	5	2	1																																		
PS									1		1	1	2	1	1	1	1	2	1	2	2	2	2	2	1	2	1	1	1	1	2	1	1	1	1	1	1	1	1		3
SN	12	12	9	7	6	4	3	2	1	1																															
SS										1	1	1	2	1	1	1	1	1	1	2	2	2	2	1	2	2	1	1	2	1	2	1	2	1	1	1	1	1	1	1	3

Hyperplane-noise level .10. Scatter of loading magnitudes in intervals .01(j ± 1) for j = 1,3,5,... .

	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35	37	39	41	43	45	47	49	51	53	55	57	59	61	63	65	67	69	71	73	75	77	79	> 80
PN	9	8	10	8	7	5	3	2	1	1	1																														
PS										1	1	1	1	1	1	1	2	1	2	2	3	2	2	2	2	2	2	2	1	2	1	1	1	1	1	1	1	1	1		2
SN	9	8	7	7	6	5	4	3	2	1	1	1																													
SS										1	1		1	1	2	1	1	2	2	2	2	2	2	2	2	2	2	1	2	1	1	1	1	1	1	1	1				2

Hyperplane-noise level .15. Scatter of loading magnitudes in intervals .01(j ± 1) for j = 1,3,5,... .

	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35	37	39	41	43	45	47	49	51	53	55	57	59	61	63	65	67	69	71	73	75	77	79	> 80
PN	7	7	7	7	8	5	4	3	2	1	1	1	1	1	1																										
PS										1	1	1	1	1	1	2	1	2	2	2	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1		1	1	1	1	2
SN	6	7	7	7	6	5	4	3	2	2	1	1	1	1	1																										
SS									1	1	1	1	1	2	1	2	1	2	1	2	1	1	2	2	2	2	2	2	2	1	1	1	1	1	1	1		1	1		3

Hyperplane-noise level .20. Scatter of loading magnitudes in intervals .01(j ± 1) for j = 1,3,5,... .

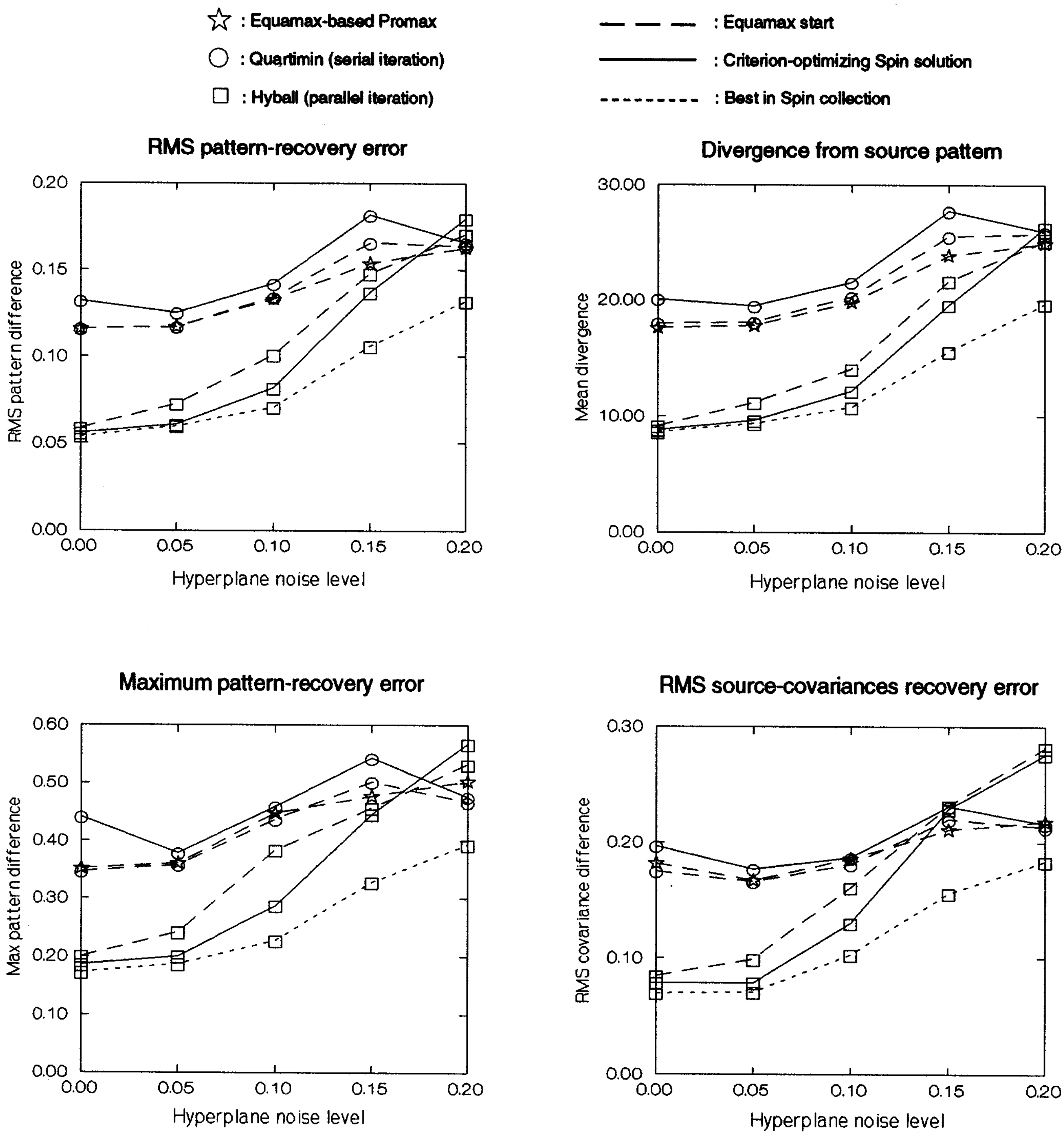

Figure 1. Source-recovery error by selected rotation methods as a function of hyperplane noise. The Spin curves for Promax are indistinguishable from the Equamax-start Promax results shown here.

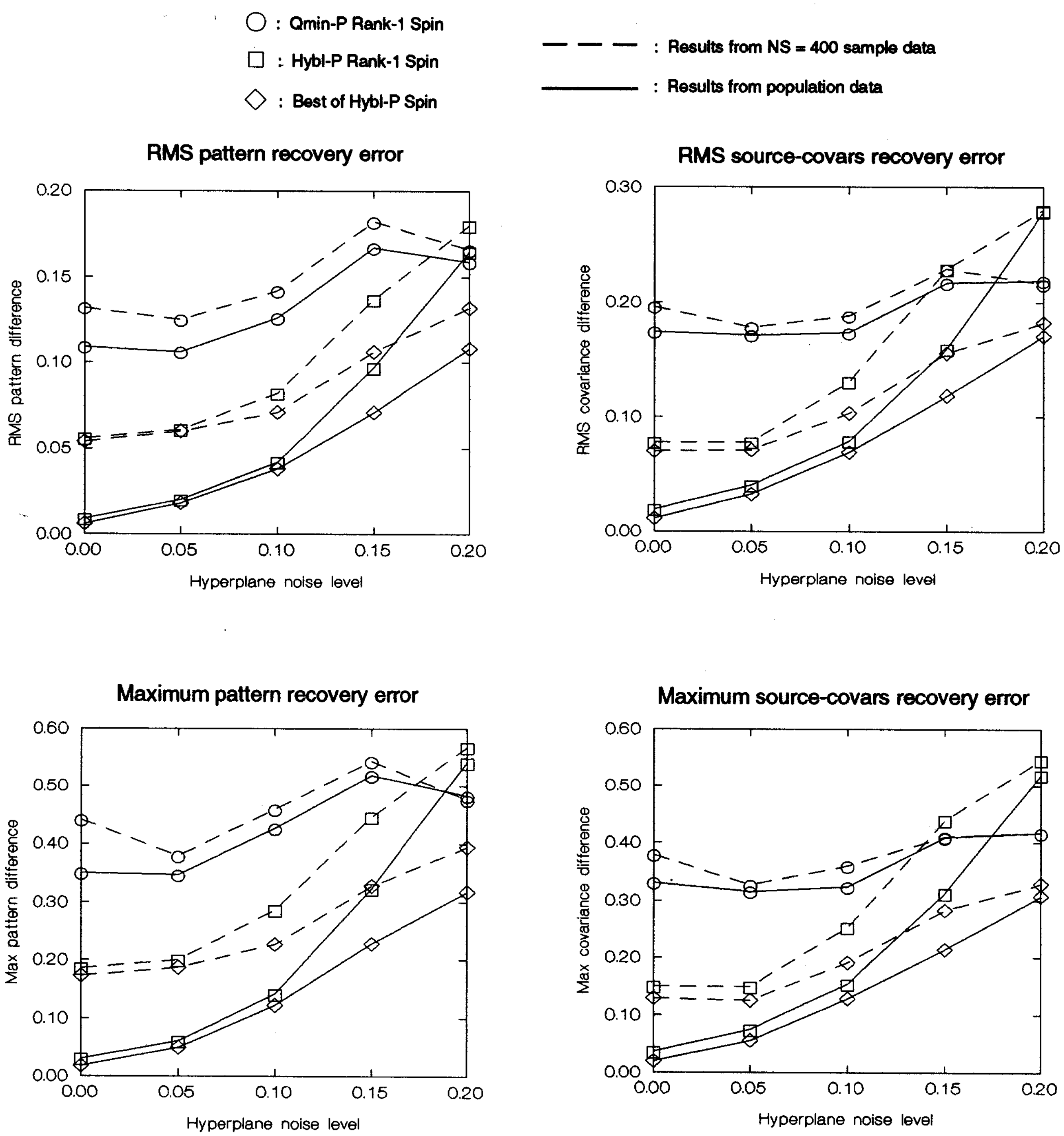

Figure 2. Improvement in source recovery when the sample covariances of items factored in this study are replaced by their population values.