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Hyball: A Method for Subspace-Constrained Factor Rotation

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Want to rotate the latent variables you have found by some method of factor extraction to oblique simple structure while leaving invariant selected axes or subspaces in your initial solution? Here's how.

So much has been published on the methodology of factor rotation, and so many computer programs released for simple-structure solutions by one criterion or another, that one might well favor a moratorium on this topic. Yet on the side of theory, the approach to oblique rotation pioneered by Maxplane (Cattell & Muerle, 1960; Eber, 1966) and Functionplane (Katz & Rohlf, 1974) — analytic emulation of subjective rotation to densest hyperplanes — has remained largely neglected. And more importantly, on the side of practice, the routines for factor rotation available in commercial data-analysis libraries (IMSL) or software packages (SPSS, SAS, BDMP, SYSTAT, etc.) on which most users must rely are obdurately inflexible in how they can be applied. So it should please you to learn that the customized control over rotation that deep down inside you have always wanted can now be yours. Specifically, I would like to acquaint you with HYBALL, a program for oblique rotation with distinctive versatilities not currently available elsewhere. While the most important of these features can easily be included in future editions of any other program whose rotation is direct, HYBALL can be running on your microcomputer tomorrow.

Although HYBALL — short for *hyperplane eyeballing* — is named for its emulation of subjective rotation, its main motivation is to allow invariance of selected factor axes or subspaces while others become oblique to them. So first of all I had better explain why this is a Good Thing. Briefly, the reason is that initial factoring sometimes manages to position certain of its axes where we think they should be, at least up to rotation within restricted blocks thereof, so that we want these axes or subspaces to persist in our terminal solution. To illustrate, let me introduce the example that will later be detailed numerically.

Data released by the Psychological Corp. (Wechsler, 1981) for the sample population on which the WAIS-R was standardized include subject scores not merely on the 11 WAIS-R subscales, but also on several background variables that in Experimental-design conceptions of multivariate relations would be viewed as random-effects *treatment* factors. Two of these, chronological age (Age) and education level (Educ), show substantial correlations with the WAIS-R subscales; and the question thus arises, when factoring the WAIS-R data for disclosure of common sources, how can we best include Age and Educ in the analysis? There are various ways to approach this, one being simply to treat Age and Educ as two more output measures to be factored jointly with the others. However, common sense urges us to view Age and Educ, or at least variables of which these are distinguished indicators, as causal influences on success at intelligence tasks rather than additional effects of the latter's proximal causes. So letting column vector $Y = (y_1, ..., y_{11})$ comprise the 11 WAIS-R subscales while $X = (x_1, x_2) =$ (Age, Educ) is the column vector of manifest input variables, it is more reasonable to begin by inquiring whether the covariances among performance variables Y might be due just to their mutual dependence on X.

For simplicity treating measures X as errorless, we can test this initial hypothesis by partialling X out of Y and observing whether the residual Y-covariances vanish — which they most emphatically do not in this case. Insomuch as there are appreciable relations among the WAIS-R subscales prima facie unaccounted for just by their common sources Age and Educ, we next turn to models positing that the WAIS-R subscales are determined not only by X but additionally by some number r of latent common factors $F = (f_1, ..., f_r)$ as well. That is, we now want to solve for pattern matrices **A**, **B**, and diagonal **D** in

$$Y = \mathbf{A}F + \mathbf{B}X + \mathbf{D}U \qquad (1)$$

that reproduce from F and X all the Y-covariances except residual variances attributable to unique factors U.

An initial solution for the coefficients in Equation 1 is entirely straightforward: We simply partial X out of Y, take the regression coefficients of Y upon X for **B**, and extract **A** from the residual Y-covariance matrix $\mathbf{C}_{YY.X}$ by whatever common-factoring algorithm we favor, say iterated principal axes. But we cannot cogently interpret the pattern of Y on these initial (F, X)-axes as structural weights unless we argue that Y's proximal latent sources are uncorrelated with Age and Educ. Far more plausible is that whatever latent factors most immediately determine WAIS-R success along with Age and Educ are themselves importantly influenced by the latter, and indeed may mediate most if not all of the latter's effects upon the former. That is, the reason why Age and Educ influence WAIS-R responding may well be that performance on intelligence tests is due to certain mental abilities which develop in part as a function of maturation and training. Starting from our initial solution for the pattern on Equation 1 with F orthogonal to X, we can appraise this causal-mediation prospect by searching for an alternative r-tuple $G = \mathbf{W}_1F + \mathbf{W}_2X$ of axes in combined (F, X)-space such that Y's rotated pattern coefficients on X in

$$Y = (\mathbf{A}\mathbf{W}_1^{-1})G + (\mathbf{B} - \mathbf{A}\mathbf{W}_1^{-1}\mathbf{W}_2)X \quad [F = \mathbf{W}_1^{-1}(G - \mathbf{W}_2X)]$$

are as close to zero as choice of $\mathbf{W}_2$ can achieve. And this, in turn, can be nicely accomplished by rotating the joint pattern [**A B**] of Y on (F, X) in Equation 1 to simple structure under the constraint that the X-axes remain invariant. You will see later for the WAIS-R data just how striking the results so obtained can be.

The rationale of this mediation model can be relaxed in various ways that call for more complex rotational constraints. For example, we might conjecture (nevermind how plausibly) that although Age and Educ lie in a 2-dimensional space of relatively remote WAIS-R sources, the causal basis of this space may not align perfectly with Age and Educ. In that case, we want to rotate all of factors (F, X) in Equation 1 to simple structure under the constraint that the subspace spanned by X remains invariant even though its rotated axes need not coincide with Age and Educ. Alternatively, if we hypothesize that Age is a causal source of WAIS-R and that a second one is also in X-space even though it may not be collinear with Educ, we want a hierarchy of subspace constraints on rotation of Equation 1 under which the invariances are to be first of all X-space and, nested within that, its one-dimensional subspace spanned by Age.

Finally, note that the factor axes/subspaces we want rotation to leave unaltered needn't consist just of data variables. For example, were we to obtain reliability coefficients showing that recorded Age and Educ are appreciably contaminated by measurement error, we could replace these measures' observed variances by their estimated true variances and proceed to analyze the WAIS-R covariance structure exactly as before except that X in Equation 1 would now comprise the true-parts of Age and Educ. More broadly, whenever a complex structural model describable by a path diagram on blocks of latent variables imposes arbitrary constraints such as triangularity of pattern or orthogonality of covariances on its within-block parameter arrays in order to specify a determinate solution, it is appropriate to remove these by a subsequent rotation that searches for simple structure (or any other pattern ideal you may prefer) under invariance of the latent axes/subspaces whose positionings are *not* arbitrary in the model. How often need for subspace-constrained rotation arises in structural modeling practice, I have no idea. But HYBALL can impose a complex nested hierarchy of subspace invariances just as easily as it can fix single axes.

The Algebra of Subspace Invariance Under Factor Rotation

HYBALL finds simple structure by iterating single-plane pattern shifts (direct rotation) while achieving invariance of selected factor axes/subspaces through an appropriate layout of zeros imposed on the rotation matrix.

Computationally, this is so elementary that most of HYBALL's technicalities which need to be put on record can be relegated to the Appendix. But the algebra of subspace-invariant rotation merits an explicit theoretical statement.

Let $Y = (y_1, ..., y_n)$ comprise common parts of our data variables for which we have developed an initial factoring

$$(2) \qquad Y = \mathbf{A}F, \quad \mathbf{C}_{YY} = \mathbf{A}\mathbf{C}_{FF}\mathbf{A}'$$

for some $\mathbf{A}$ and $\mathbf{C}_{FF}$. (Decomposition Equation 2 is the relevant fragment of some larger decomposition that also generally includes uniquenesses and reproduction residuals. Factor pattern $\mathbf{A}$ and factor covariances $\mathbf{C}_{FF}$ are presumed to be numerically identified, whereas subjects' scores on F are typically unknown.) Then for any nonsingular but otherwise arbitrary rotation $G = \mathbf{T}F$ of the F-axes, the G-covariances and the pattern of Y on G are given by

$$(3) \qquad Y = (\mathbf{A}\mathbf{T}^{-1})G, \quad \mathbf{C}_{GG} = \mathbf{T}\mathbf{G}_{FF}\mathbf{T}' \quad (G = \mathbf{T}F).$$

If $\mathbf{T}$ in Equation 3 fails to normalize the G-covariances, we can subsequently achieve this by putting $\mathbf{D}_v =_{\text{def}} [\text{Diag}(\mathbf{C}_{GG})]^{-½}$ and rescaling both the G-axes and the pattern on G in accord with $Y = (\mathbf{A}\mathbf{T}^{-1}\mathbf{D}_v^{-1})(\mathbf{D}_v G)$. Entrenched orthodoxies notwithstanding, HYBALL finds it most theoretically insightful and computationally efficient to let obliquely rotated factors have arbitrary variances in our basic formulas, adding variance normalization only after the rotation's essence is clear.

Suppose, now, that with factor totalities F and G partitioned as $F = (F_1, F_2)$ and $G = (G_1, G_2)$ with $G_1 (G_2)$ having the same dimensionality as $F_{11} (F_2)$, rotation by Equation 3 is constrained to have the subspace spanned by F_2 also spanned by G_2. This is equivalent to requiring $G_2 = \mathbf{V}_2 F_2$ for some nonsingular $\mathbf{V}_2$, where $\mathbf{V}_2$ may be under additional constraints (possibly as extreme as axis fixation $G_2 = F_2$) to which we are at present indifferent. Then rotation matrix $\mathbf{T}$ and its inverse are restricted to block-triangular form

$$(4) \qquad \mathbf{T} = \begin{bmatrix} \mathbf{V}_1 & \mathbf{W} \\ \mathbf{0} & \mathbf{V}_2 \end{bmatrix}, \mathbf{T}^{-1} = \begin{bmatrix} \mathbf{V}_1^{-1} & -\mathbf{Q} \\ \mathbf{0} & \mathbf{V}_2^{-1} \end{bmatrix} \qquad (\mathbf{Q} = \mathbf{V}_1^{-1}\mathbf{W}\mathbf{V}_2^{-1}),$$

corresponding to a subspace-constrained rotation

$$(5) \qquad \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} = \begin{bmatrix} \mathbf{V}_1 & \mathbf{W} \\ \mathbf{0} & \mathbf{V}_2 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} \mathbf{V}_1 F_1 + \mathbf{W}F_2 \\ \mathbf{V}_2 F_2 \end{bmatrix}.$$

When $\mathbf{A}_1$ and $\mathbf{A}_2$ are the subpatterns in $\mathbf{A} = [\mathbf{A}_1\ \mathbf{A}_2]$ on F_1 and F_2, respectively, the factor pattern that results from rotation Equation 5 has algebraic composition

$$(6) \qquad \mathbf{AT}^{-1} = (\mathbf{A}_1\ \mathbf{A}_2)\begin{bmatrix} \mathbf{V}_1^{-1} & -\mathbf{Q} \\ \mathbf{0} & \mathbf{V}_2^{-1} \end{bmatrix} = [\mathbf{A}_1\mathbf{V}_2^{-1}\quad (\mathbf{A}_2\mathbf{V}_2^{-1} - \mathbf{A}_1\mathbf{Q})].$$

Moreover, although $\mathbf{V}_1$ and $\mathbf{V}_2$ in Equation 4 to Equation 6 must be nonsingular, $\mathbf{T}$'s off-diagonal submatrix $\mathbf{W}$ or its counterpart $-\mathbf{Q}$ in $\mathbf{T}^{-1}$ remains entirely unconstrained. So if we solve for $\mathbf{Q}$ to maximize the number of near-zero elements in $(\mathbf{A}_2\mathbf{V}_2^{-1} - \mathbf{A}_1\mathbf{Q})$ given our preferences for $\mathbf{V}_1$ and $\mathbf{V}_2$, Equation 5 yields a rotation of $F = (F_1, F_2)$ into $G = (G_1, G_2)$ wherein G_2 spans the same subspace as F_2 (in fact, we can have $G_2 = F_2$ by taking $\mathbf{V}_2 = \mathbf{I}$), while the repositioning of F_1 as G_1 in joint (F_1, F_2)-space minimizes the putative effects of G_2 on Y unmediated by G_1.

What has been shown is that if F_b is any restricted block of the factor axes in $Y = \mathbf{A}F$, a necessary and sufficient condition for invariance of the subspace spanned by F_b under rotation $G = \mathbf{T}F$ is for a suitable permutation of $\mathbf{T}$'s rows and columns — namely, one that puts F_b at the end of the permuted F and the corresponding G_b at the end of the permuted G — to be block-triangular as in Equation 5 with F_b for F_1 and G_b for G_1. $\mathbf{T}$ may have many different permutations into such block-triangularity (cf. the limiting case of diagonal $\mathbf{T}$), so it can leave a multiplicity of factor subspaces invariant, some perhaps nested in others. HYBALL exploits this implicit block-triangularity principle by imposing on $\mathbf{T}$ a fixed assignment of zero elements describable by a *rotation control* matrix $\mathbf{K}$ of the same order as $\mathbf{T}$ and containing just zeros and unities. If the *ij*th element $\mathbf{K}_{ij}$ of $\mathbf{K}$ is zero, $\mathbf{T}_{ij}$ is required to be zero; whereas if $\mathbf{K}_{ij} = 1$, $\mathbf{T}_{ij}$ is allowed to be whatever our criterion for rotated axis positioning may elect at any stage of the solution iteration. Let us say that rotation matrix $\mathbf{T}$ is *K-structured* just in case every zero element of $\mathbf{K}$ is also zero in $\mathbf{T}$. And say also that axis f_i in $F = (f_1, ..., f_r)$ is *(rotationally) vulnerable* to axis f_j under $\mathbf{K}$-structured rotation iff $(\mathbf{K})_{ij} = 1$, that is, just in case $\mathbf{K}$ allows f_i to shifted into g_i by adding to f_i some weighting of f_j. Clearly, any g_i produced by rotation $G = \mathbf{T}F$ lies in the subspace spanned just by the F-axes to which f_i is vulnerable. So for any block of factors F_J in F indexed by some subset J of integers from 1 to r, if the axes in F_J are rotationally vulnerable only to ones that are themselves in F_J, block G_J of the rotated factors will span the same subspace as F_J.

In order for control matrix $\mathbf{K}$ to impose on $\mathbf{K}$-structured $\mathbf{T}$ the implicit block-triangularities that yield subspace invariances as wanted, however, the rotation-vulnerability relation defined by $\mathbf{K}$ must be *transitive* in the sense that $\mathbf{K}_{ik} = 1$ whenever both $\mathbf{K}_{ij} = 1$ and $\mathbf{K}_{jk} = 1$ for any j. It is straightforward to show that $\mathbf{K}$

is transitive just in case the factor indices can be partitioned into disjoint subsets $J_1, ..., J_s$ $(1 \leq s \leq r)$ in such fashion that the blocks $F_{J_1}, ..., F_{J_s}$ of F-axes respectively picked out by these J-blocks are *partially ordered* by **K**-controlled rotation vulnerability as follows: (a) If f_i is vulnerable to f_j, then every f_h in the same J-block as f_i is vulnerable to every f_k in the same J-block as f_j; and (b), if f_i is vulnerable to f_j, then f_j is *not* vulnerable to f_i unless f_i and f_j are in the same J-block. From there, it follows for any transitive **K** that if **T** is **K**-structured so is $\mathbf{T}^{-1}$, and if $\mathbf{T}_1$ and $\mathbf{T}_2$ are both **K**-structured so is $\mathbf{T}_1\mathbf{T}_2$. (Hence iteration of **K**-structured rotation preserves structure **K**.) Moreover, if sequence $J_1, ..., J_s$ of J-blocks reflects the partial-ordering by **K** in the sense that the axes in $\mathbf{F}_{J_i}$ are vulnerable to the axes in F_{J_k} only if $i < k$, then for each $k = 1, ..., s$, the subspace spanned jointly by axis blocks $F_{J_k}, F_{J_{k+1}}, ..., F_{J_s}$ is invariant under **K**-structured rotation. (Note that because this is only a *partial* ordering, there is generally more than one sequence of J-blocks having this property.)

Sketch of Proof

If rotation matrix **T** is **K**-structured for transitive **K**, we can always choose the order $J_1, ..., J_s$ of its index J-blocks to reflect **K**'s partial ordering of factor blocks in the sense just described while ordering F as $F = (F_{J_1}, F_{J_2}, ..., F_{J_s})$. Then G and the rows of **T** can be ordered to give **T** block-triangular form

$$\begin{bmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} & \cdots\cdots & \mathbf{T}_{1s} \\ \mathbf{0} & \mathbf{T}_{22} & \cdots\cdots & \mathbf{T}_{2s} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \cdots \;\; \mathbf{0} & \mathbf{T}_{ss} \end{bmatrix}$$

with $\mathbf{T}_{ik}$ the coefficients of F_{J_k}'s contribution to G_{J_i}. Each $\mathbf{T}_{ii}$ on the diagonal therein must be nonsingular if **T** is to be a full-rank rotation, but any $\mathbf{T}_{ik}$ $(i < k)$ can be zero either by solution fortuity or by **K**-stipulation. From here, proof of the claims made above is routine.

The Elegance of Single-axis Pattern Shifts

Selecting a transitive control matrix **K** is only half the problem for subspace-constrained factor rotation. Also necessary, obviously, is to devise a method for assigning numerical values to whatever coefficients in **T** are allowed by **K** to be nonzero. Alternatives for this are determined first of all by what sort of pattern ideal is the rotation's target, and secondly by details of how that is to be approached. HYBALL goes about the latter by iterating aggregates of Equation 4 rotations for varied choices of the to-be-shifted axis block F_1. And dexterity is optimized by taking F_1 to be a singleton while F_2 is temporarily fixed.

To clarify, consider the special case wherein only one axis is moved, say f_1. That is, put $F_1 = (f_1)$ and $\mathbf{V}_2 = \mathbf{I}$ in Equation 4. With no further loss of generality we can also choose $\mathbf{V}_1 = 1$, whereupon Equation 6 simplifies to

$$(7)\quad Y = [\mathbf{a}_1(\mathbf{A}_1 - \mathbf{a}_1\mathbf{w})]\begin{bmatrix} g_1 \\ G_1 \end{bmatrix}\quad \left(Y = [\mathbf{a}_1, \mathbf{A}_2]\begin{bmatrix} f_1 \\ F_2 \end{bmatrix},\ \begin{bmatrix} g_1 \\ G_2 \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{w} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}\begin{bmatrix} f_1 \\ F_2 \end{bmatrix}\right),$$

in which $\mathbf{a}_1$ is the first column of Y's unrotated pattern $\mathbf{A}$ on F [$= (f_1, F_2)$], $\mathbf{A}_2$ comprises the last r-1 columns of $\mathbf{A}$, $\mathbf{w}$ is a $1 \times (r\text{-}1)$ row of rotation coefficients, and g_1 is f_1 plus a component $\mathbf{w}F_2$ in a possibly-restricted part of F_2-space. (That is, $\mathbf{K}$ may require some elements of $\mathbf{w}$ to be zero.) Since $G_2 = F_2$, f_1 is the only axis repositioned by this rotation. Let $\mathbf{a}_j$ be the jth column of $\mathbf{A}$, that is, the (j-1)th column of $\mathbf{A}_2$, while w_j is the (j-1)th element of $\mathbf{w}$. Then rotation Equation 7 just of axis f_1 leaves the first-factor pattern coefficients $\mathbf{a}_1$ unchanged (though subsequent normalization of g_1's variance will rescale $\mathbf{a}_1$), while for each $j = 2, ..., r$, the column of Y's pattern on the jth axis f_1 changes from $\mathbf{a}_1$ to

$$(8)\qquad \tilde{\mathbf{a}}_j = \mathbf{a}_j - w_j\mathbf{a}_1,$$

where $\tilde{\mathbf{a}}_j$ is the jth column of the rotated pattern. (Note that choice of w_j affects *only* the jth pattern column, which is to say that these planar shifts of f_1 do not interact.)

Equation 8 corresponds to a rotation of factor f_1 just in the f_1 / f_j plane, and can easily be understood graphically.[1] Consider the illustration in Figure 1 (next page) of the scatter plot of initial factor loadings ($\mathbf{a}_1$, $\mathbf{a}_j$) in the f_1/f_j plane, each point i therein showing the joint entries in columns 1 and j of $\mathbf{A}$ for the item (output variable) y_i whose factor loadings are given by $\mathbf{A}$'s ith row. Equation 8 is simply the equation for the line g_1 through this graph's origin whose angle to f_1 is arctan(w_j).[2] And any item's loading on f_j (= g_j) after Equation 8 rotation of f_1 is just the item's vertical distance from line g_1, whence in particular, any item whose f_1/f_j point lies on g_1 has a post-rotation loading on f_1 of zero. So g_1 is the intersection of the f_1/f_j plane with the post-rotation hyperplane to f_j. And graphic line g_1 also represents rotated factor g_1 in that the g_1 / f_j pattern plot is what we

[1] Versions of Equation 8 have previously been published by Jennrich & Sampson (1966, p. 316) and Mulaik (1972, p. 282); but their explicit inclusion of a coefficient normalizing g_1's variance has obscured recognition how *perspicuous* a graph like Figure 1 can make the nature of single-plane direct rotation.

[2] This is the angle between lines f_1 and g_1 in the graph plotting f_j against f_1 at a right angle regardless of the factor correlation, not the angle whose cosine is the correlation between f_1 and g_1. (Factor correlations enter direct rotation only in variance renormalizations.)

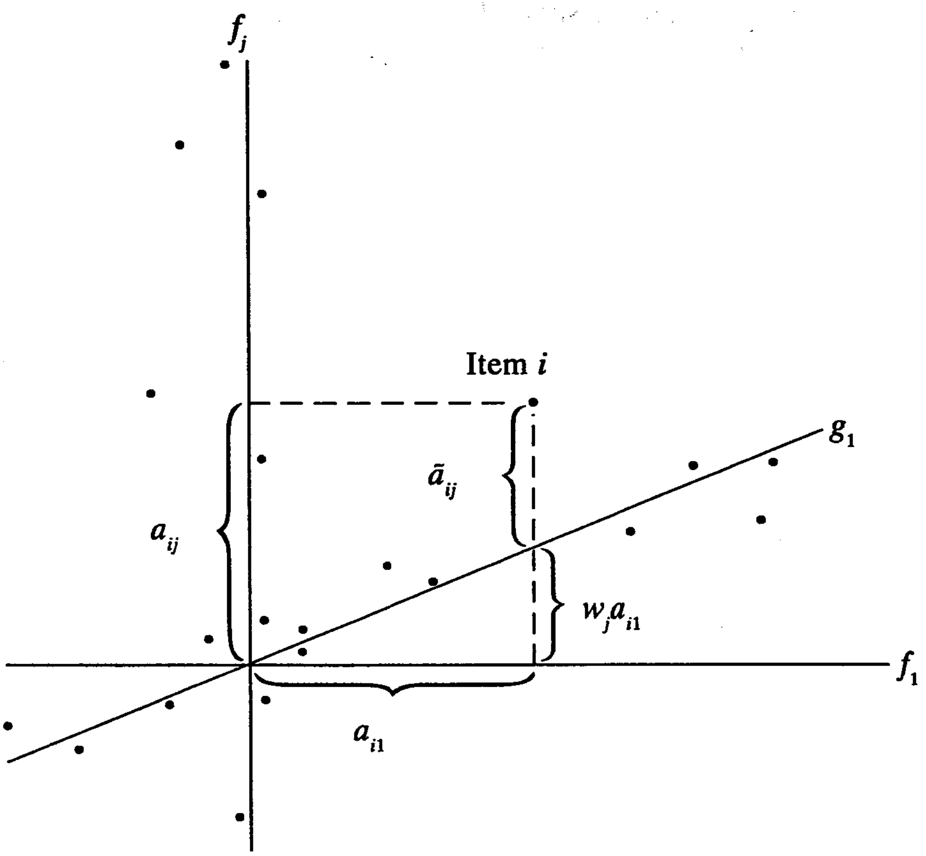

Figure 1
Graphic representation of Equation 8. g_1 is the line in the f_1/f_j plane whose slope relative to the f_1-axis is w_j.

get by pushing line g_1 down to horizontal while similarly displacing all the item points downward to preserve their vertical distances from g_1 and horizontal distances from f_j.

Choosing w_j in Figure 1 is the standard move of graphic rotation except for working directly with item loadings on primary factors rather than with item correlations with reference axes. For any planar pattern ideal that we can effectively describe — it needn't be simple structure, though that is HYBALL's aim — it is relatively straightforward to devise a measure of how well any given pair ($\mathbf{a}_i$, $\mathbf{a}_j$) of pattern columns approaches this ideal, and from there to program a function on pairs of pattern columns whose output for any planar pattern ($\mathbf{a}_i$, $\mathbf{a}_j$) is the coefficient w_{ij} that optimizes the pattern ($\mathbf{a}_i$, $\tilde{\mathbf{a}}_j$) into which ($\mathbf{a}_i$, $\mathbf{a}_j$) is rotated by Equation 8 axis shift $\tilde{\mathbf{a}}_j = \mathbf{a}_j - w_{ij}\mathbf{a}_i$. In particular, for every (i, j) such that $\mathbf{K}_{ij} = 1$, HYBALL searches out a line through the f_i / f_j plane's origin (not so close to f_j as to threaten factor collapse) that pattern points lying close thereto demark with special prominence (again, see Figure 1), and outputs the w_{ij} that rotates f_i to this position. Iteration of the $\mathbf{K}$-allowed planar shifts so computed then yields HYBALL's global solution for simple structure.

Other HYBALL Felicities

There are many analytic criteria by which we can search factor planes for simple structure. Elsewhere (Rozeboom, 1991) I have discussed this matter in some detail, with emphasis on a broad class of efficiently computable hyperplane-fit measures that includes or well-approximates most that we might seriously wish to use. Any member of this class (including the Quartimin criterion, though that is not recommended) is available to HYBALL users through choice of control parameters. Most of HYBALL's remaining technicalities, notably how planar shifts are iterated to a global solution, need not detain you though they are given in the Appendix if you care. You should, however, be apprised that HYBALL is an interactive program whose flexibility should by rights make the method of choice for difficult rotations even when free of subspace constraints. Not merely can many variations of the control parameters be tested at one sitting, a record is kept of all provisional solutions and any can be recalled at any time with guidance by a screen display of hyperplane counts for all the stored solutions. Plane plots of the currently active pattern are also available on screen; and in light of these the user can fine-tune the rotation's continuation by instructing it to ignore selected item points in selected planes.

An Empirical Example

What HYBALL can accomplish through subspace constraints is illustrated in Figure 2 (next page). The data analyzed are scores of 1880 young-to-mature adults on the 11 subscales of the revised Wechsler Adult Intelligence Scale together with subjects' ages and education levels. Both Age and Educ show substantial correlations with the WAIS-R subscales (Educ with all; Age with just the nonverbal subscales), and can be viewed as causal sources of the latter or at least as (let us pretend) nearly-errorless indicators of certain dimensions in WAIS-R source space. But orthodox common-factoring of the WAIS-R correlations also implicates three or four latent sources which should themselves be dependent on Age and Educ if distinct from them. So the question naturally arises: Do Age and Educ discernably influence WAIS-R competence only through the mediation of their effects on more imminent common factors of the WAIS-R, or do the former appear also to have WAIS-R effects along causal paths independent of the latter?

To answer, we first partial Age & Educ out of the variance-normalized WAIS-R subscales, and then extract orthonormal common factors from the

HYBALL ROTATION OF MODA PATTERN FOR THE WAIS-R SUBSCALES ON THREE LATENT AND TWO MANIFEST COMMON SOURCES.

The input pattern of 11 WAIS-R subscales on 5 factors (F1 = Age, F2 = Education; communalities in parentheses) is

1.	(.76)	.15	.60	.59	-.23	-.08	(Information)
2.	(.49)	-.06	.42	.45	-.05	.33	(Digit Span)
3.	(.87)	.18	.62	.64	-.29	-.07	(Vocabulary)
4.	(.69)	.06	.53	.58	-.04	.28	(Arithmetic)
5.	(.69)	.10	.56	.59	-.17	-.10	(Comprehension)
6.	(.71)	-.09	.54	.61	-.12	-.11	(Similarities)
7.	(.66)	-.27	.40	.60	.17	-.12	(Picture Completion)
8.	(.56)	-.33	.38	.52	.04	-.05	(Picture Arrangement)
9.	(.72)	-.31	.39	.59	.30	.04	(Block Design)
10.	(.65)	-.28	.31	.55	.38	-.09	(Object Assembly)
11.	(.58)	-.45	.40	.40	.07	.05	(Digit Symbol)
12.	(1.00)	1.00	.00	.00	.00	.00	(Age)
13.	(1.00)	.00	1.00	.00	.00	.00	(Education)

with corresponding factor covariances

1.00	-.15	.00	.00	.00
-.15	1.00	.00	.00	.00
.00	.00	1.00	.00	.00
.00	.00	.00	1.00	.00
.00	.00	.00	.00	1.00

(The first 2 factors are manifest input variables.)

Control matrix for this rotation, with KTL(I,J) = 1 signifying that Factor J rotates Factor I, was:

1	0	0	0	0
0	1	0	0	0
1	1	1	1	1
1	1	1	1	1
1	1	1	1	1

This pattern was transformed by factor-rotation matrix

1.00	.00	.00	.00	.00
.00	1.00	.00	.00	.00
.10	.67	.70	-.24	-.10
-.43	.35	.65	.46	-.11
-.01	.62	.67	-.06	.40

into rotated factor pattern

1.	(.76)	.04	.03	.87	-.05	.02	(Information)
2.	(.49)	-.06	-.01	-.07	-.05	.79	(Digit Span)
3.	(.87)	.03	-.02	.97	-.11	.05	(Vocabulary)
4.	(.69)	.08	.00	.08	.05	.74	(Arithmetic)
5.	(.69)	.05	.02	.82	.06	-.04	(Comprehension)
6.	(.71)	-.11	.00	.77	.14	-.04	(Similarities)
7.	(.66)	-.06	-.02	.37	.56	-.04	(Picture Completion)
8.	(.56)	-.24	-.03	.40	.30	.06	(Picture Arrangement)
9.	(.72)	-.02	.01	-.04	.67	.27	(Block Design)
10.	(.65)	.07	.01	.01	.82	.01	(Object Assembly)
11.	(.58)	-.36	.09	.13	.24	.22	(Digit Symbol)
12.	(1.00)	1.00	.00	.00	.00	.00	(Age)
13.	(1.00)	.00	1.00	.00	.00	.00	(Education)

with corresponding factor covariances

1.00	-.15	.00	-.48	-.10
-.15	1.00	.65	.42	.62
.00	.65	1.00	.59	.85
-.48	.42	.59	1.00	.63
-.10	.62	.85	.63	1.00

Figure 2

Reproduction of HYBALL's output file (sans pattern-plane graphs) for rotation of the WAIS-R factor pattern with Age & Education fixed as manifest sources. Before printing, a text-editor was used to insert the WAIS-R subscale names, expand some legends, and delete some superfluous information.

latter's residual covariances either by iterated principal factoring (which produced the pattern rotated in Figure 2) or by any other factoring method if preferred. For the present data, both scree considerations and tidiness of the ensuing results urge that the proper number of common residual factors is three. The WAIS-R loadings on these are shown in the last three columns of Figure 2's initial pattern, while this pattern's first two columns comprise the joint regression weights of Age & Educ for the WAIS-R subscales.[3]

[3] Although details of this initial factor solution don't really matter here, salient technicalities are as follows: (a) The first eight eigenvalues of the WAIS-R zero-order correlation matrix are 6.49, 1.17, .67, .54, .43, .39, .33, .29, having a weak scree break after the 4th. After Age & Educ are partialled out, the first six eigenvalues of the residual covariances are 3.75, .75, .66, .45, .42, .38, with a conspicuous scree break after the 3rd. (b) The proportion of total WAIS-R normalized variance accounted for jointly by Age, Educ, and $r = 0, 1, \ldots, 6$ latent factors (iterated principal factoring) are respectively .298, .437, .500, .536, .556, .569, .585. (c) Age, Educ, and $r = 3$ latent factors jointly reproduce the WAIS-R correlations with RMS residual error of .012 and maximum error of .033.

Following Varimax pre-rotation of the three latent factors within their initial subspace (an optional commencement of HYBALL), rotation of the input pattern with Age & Educ fixed converged in six iteration steps to the simple-structure pattern shown at bottom in Figure 2. The hyperplanes revealed there are exceptionally clean, and that HYBALL can find these attests to its proficiency. But other methods of oblique rotation could also do more or less this well were they to include provision for subspace constraints. Where HYBALL displays its distinctive competence is in the rotated pattern coefficients on fixed factors Age & Educ. The weights on Educ, which were so prominent in the initial factor pattern, have all vanished, strongly supporting the hypothesis that education has no bearing on WAIS-R competence except through the three abilities factors (you name them) diagnosed by columns 3-5 of the rotated pattern. In contrast, although the same is also largely true of Age, variables 8 and 11 (Picture Arrangement and Digit Symbol) are exceptions conspicuous enough to merit further inquiry by WAIS-R studies. I am not myself inclined to make much of factorial results from any array of variables so small as this one. But you must agree that the pattern and associated factor correlations disclosed by HYBALL in this case are strongly *provocative*.

Program Availability

FORTRAN-77 source code for HYBALL, together with its compilation for DOS execution on AT-compatible machines either with or without a math coprocessor, are available on request in a package that also contains .FOR and .EXE code for several other programs that facilitate HYBALL's use. These include computation of data covariances from raw-score files, construction of item scales to estimate the HYBALL factors, and, most importantly, initial factor extraction by program MODA (Multiple-output Dependency Analysis). Given a covariance matrix as input, MODA allows selection of zero or more of the received variables to be manifest sources (the *X*-set), selects some or all of the remainder to be outputs (the *Y*-set), computes the *Y*-set's regression upon the *X*-set; factors the latter's residuals either by iterated principal-factoring or by Minres for each of the common-factor dimensionalities in a range chosen by the user in light of the residual *Y*-covariance eigenvalues, and files these initial patterns for transmission to HYBALL. (The pattern initiating Figure 2 was produced by MODA.) Send me $10 for expenses, and four 5¼ inch or two 3.5 inch floppies bearing these and other goodies developed more recently, together with instructions for their installation and operation will be yours by return mail. Or if your budget is tight, write me anyway for a freebie. Unless you advise me otherwise, I'll presume that you want .EXE code that requires a math coprocessor. Source code for UNIX is also available.

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Appendix

The assorted local tactics whose aggregate constitutes the HYBALL program are of three sorts. First is stipulation of whatever subspace invariances may be wanted. In principle, this requires the user to enter rotation-control matrix **K**; but patterns received by HYBALL from its support programs include default specifications of K derived from higher-level decisions made previously, notably, choice of an *X*-set in MODA or imposition of a causal-path structure on blocks of the MODA factors by a program (HYBLOCK) whose rationale must await another occasion,[4] that seldom need to be overridden. Even so, it is relatively simple to reset **K** by following on-screen instructions for this.

Next comes selecting a criterion for the single-plane axis shifts illustrated in Figure 1. HYBALL provides a broad repertoire of *misfit* functions ϕ for optimizing these. Specifically, $\phi(\mathbf{a}_i, \mathbf{a}_j, w)$ measures how diffuse in the (f_i, f_j) plane is the hyperplane demarked by the pattern column $\tilde{\mathbf{a}}_j = \mathbf{a}_j - w\mathbf{a}_i$ resulting from factor shift $g_i = f_i + wf_j$ for arbitrary w, the best shift of f_i in this plane by this criterion being the one that minimizes ϕ. Given $(\mathbf{a}_i, \mathbf{a}_j)$, $\phi(\mathbf{a}_i, \mathbf{a}_j, w)$ is for each choice of ϕ an increasing function of the magnitude of each $\tilde{\mathbf{a}}_j$-element, but the curve of that increase differs considerably from one ϕ to another. HYBALL's parameters specifying ϕ and a method for minimizing it are detailed in Rozeboom (1991). But the program provides on-screen guidance in selection of these while encouraging default options that bypass concern for them.

Finally, given specifications of **K**, ϕ, and a solution algorithm $\phi^*(\mathbf{a}_i, \mathbf{a}_j)$ whose value for any pair $(\mathbf{a}_i, \mathbf{a}_j)$ of columns from the current factor pattern is the

[4] HYBLOCK intervenes between MODA and HYBALL, and is designed for use with data that include multiple indicators of source factors in blocks for which a causal-path structure is postulated. The program is fully operational and is documented in my forthcoming article, *HYBLOCK: A routine for exploratory factoring of block-structured data.*

approximate w that minimizes $\phi(\mathbf{a}_i, \mathbf{a}_j, w)$, we need a strategy for iterating applications of ϕ^* under constraint $\mathbf{K}$. HYBALL finds it efficient to proceed as follows: Entering any cycle of iteration with intermediate pattern $\mathbf{A} = (\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_r)$ on variance-normalized factors $F = (f_1, ..., f_r)$, the program first computes an $r \times r$ provisional-shift matrix $\mathbf{W}$ to have elements

$$\text{(9)} \qquad \mathbf{W}_{ij} = \begin{cases} \phi^*_{ij}(\mathbf{a}_i, \mathbf{a}_j) & \text{if } i \neq j \text{ and } \mathbf{K}_{ij} = 1 \\ 0 & \text{if } i = j \text{ or } \mathbf{K}_{ij} = 0 \end{cases}$$

where ϕ^*_{ij} is identical with the globally chosen ϕ^* except for ignoring the pattern points for items currently listed in OMIT for factor plane f_i / f_j. This omissions list comprises items that lie outside the zone within which axis repositioning is allowed together with any explicitly stipulated for this plane by the user, and is frequently updated. Then for any one $i = 1, ..., r$, if $\mathbf{W}_i$ is the $r \times r$ matrix whose ith row is the same as $\mathbf{W}$'s but is zero elsewhere, $G = (\mathbf{W}_i + \mathbf{I})F$ would be a $\mathbf{K}$-structured rotation of F into G that alters just axis f_i with its shift to g_i being patternwise optimal by criterion ϕ_{ij}. However, HYBALL prefers to utilize not just one row of provisional-shift matrix $\mathbf{W}$ at each rotation step but all of them.

One way to exploit $\mathbf{W}$ more fully would be to compute for each $i = 1, ..., r$ separately what pattern change would result from single-axis rotation $(\mathbf{W}_i + \mathbf{I})F$, and execute the one whose improvement is greatest. But far more economical is simply to do all the $\mathbf{W}$-approved single-axis shifts simultaneously — or rather, to reduce overshooting, to position each rotated axis g_i only partway between f_i and the ith axis in $(\mathbf{W} + \mathbf{I})F$.

To be specific, suppose that zero or more prior rotation steps have carried initial pattern $\mathbf{A}_0$ with factor covariances $\mathbf{C}_0$ into pattern $\mathbf{A}$ on variance-normalized factors F having covariances $\mathbf{C}_{FF}$. Then HYBALL's iteration step in simultaneous axis shifting is as follows: (a) Compute $\mathbf{W}$ by Equation 9; (b) Put $\mathbf{T} =_{\text{def}} (\mathbf{I} + g\mathbf{W})$ and $\mathbf{D}_v =_{\text{def}} [\text{Diag}(\mathbf{T}\mathbf{C}_{FF}\mathbf{T}')]^{-1/2}$ where g is a damping parameter in the unit interval (Values of g exceeding .5 generally seem preferable; but $g = 1$ theoretically incurs some risk of factor collapse.); (c) Execute variance-normalized rotation $G = \mathbf{D}_v\mathbf{T}F$ to obtain rotated factor pattern $\mathbf{A}_G = \mathbf{A}(\mathbf{D}_v\mathbf{T})^{-1}$ with rotated factor covariances $\mathbf{C}_{GG} = \mathbf{D}_v\mathbf{T}\mathbf{C}_{FF}\mathbf{T}'\mathbf{D}_v$; (d) Test for convergence by computing the order-r vector ε whose ith element is the angle through which the present rotation step has shifted f_i. If any term in ε is larger than a stipulated small tolerance e, say $e = .5°$, replace $\mathbf{A}$ and $\mathbf{C}_{FF}$ by $\mathbf{A}_G$ and $\mathbf{C}_{GG}$, respectively, and repeat the cycle from the beginning of (a). This iteration continues until no element of ε exceeds e or a set limit is reached, say 25 or 30 cycles. The program then records $\mathbf{A}_G$ and $\mathbf{C}_{GG}$ along with the current control parameters, and awaits user selection from a menu of continuation alternatives.

If the continuation command is to stop, the program prepares a printable file of $\mathbf{A}_0$, $\mathbf{C}_0$, $\mathbf{A}_G$, $\mathbf{C}_{GG}$, the transformation that carries F_0 into G, and, if wanted, planar plots of rotated pattern $\mathbf{A}_G$.[5] Alternatively, the user can call for screen display of $\mathbf{A}_G$ either as a numerical table or as graphic factor-plane plots, can continue rotation from the present $\mathbf{A}_G$ with or without adjustments of the control parameters, or can revert to a preceding stage of the rotation chosen in light of an accumulated record of hyperplane counts at each preceding pause. Graphic inspection of rotated pattern $\mathbf{A}_G$ allows judgment whether there are any pattern planes g_i / g_j wherein the position of moveable g_i is still subjectively suboptimal. It should also be evident for any such plane what items are degrading the solution; and continuing rotation from $\mathbf{A}_G$ with revised item omissions and perhaps other control adjustments yields a global solution more finely tuned than before.

In the event of convergence failure, ε locates the difficulty. For the indices of ε-elements larger than tolerance identify axes that have failed to converge, and in that case the program also reports to screen which axis is most wobbly in what plane. If the instability seems too large to ignore, inspecting screen displays of the item plots in troublesome planes may suggest a fine-tuning that removes the convergence obstacle.

[5] Additional options at hard-copy time are the data variables' covariances with the rotated factors, and the column multipliers for converting pattern $\mathbf{A}_G$ on the primary factors into the corresponding structure on reference axes. (The latter are useful for comparing HYBALL's performance on standard test problems to previous results for these published as reference structure.) And when a putative causal ordering is imposed on blocks of the factors by K-stipulated rotation constraints, the corresponding factor regressions are also recorded.