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Discussions

NEW MYSTERIES FOR OLD: THE TRANSFIGURATION OF MILLER'S PARADOX

Let A be the statement that a certain neutrally identified object has property a;[1] let '$P(A, X)$' denote the probability (credibility) that A is true given information X; and let 'E^a_r' abbreviate the statement '$fr(a)=r$', where $fr(a)$ is the statistical frequency (or statistical probability, or observed frequency) of property a. It is then strongly intuitive to think that for any r,

$$(1) \qquad P(A, E^a_r)=r,$$

i.e. to suppose that the probability of an object's having property a, given only the information that property a has statistical frequency r, is likewise equal to r. However, Miller (1966) and Popper (1966) have recently argued that unqualified acceptance of (1) leads to paradox, and subsequent protests by Mackie (1966) and Bub and Radner (1968) have not diminished the cogency of Miller's argument (cf. Miller 1966a, 1968). Indeed, they could not; for the argument is entirely sound. The *significance* of Miller's paradox, however, is rather different from what it seems: It does not in any way speak against (1) as a principle of inductive logic, but merely emphasises an important boundary condition which must be imposed on *any* inference schema involving hypotheticals. Even so, while Miller's own version of his paradox is indifferent to the specific content of (1), there are indeed reasons why our intuitive acceptance of the latter should not be unhesitant. For substantive paradox does in fact lurk within (1), and the spirit of this paradox remains an ominous presence even after it has been technically exorcised by suitable restrictions on (1)'s generality.

I

To develop Miller's paradox from generalisation (1), consider the hypothesis that $fr(a)=fr(\sim a)$, i.e. that $E^a_{fr(\sim a)}$ where $\sim a$ is the property of not possessing property a. It is easily proved that $fr(a)=fr(\sim a)$ if and only if $fr(a)=\cdot 5$; hence the statements $E^a_{fr(\sim a)}$ and $E^a_{\cdot 5}$ are logically equivalent. Moreover, for any two

[1] Stipulation that statement A 'neutrally identifies' the object of which it predicates a is intended to exclude cases of the sort illustrated by a's being the property 'is taller than John' while A is 'John is taller than John'. That is, we must forbid any special analytic connection between the predicate 'a' and the subject term to which a is ascribed in A.

logically equivalent statements B and B', $P(A, B)=P(A, B')$. Consequently, were (1) to hold with unlimited generality, it would follow that

$$(2) \qquad fr(\sim a)=P(A, E^{a}_{fr(\sim a)})=P(A, E^{a}_{\cdot 5})=\cdot 5,$$

which paradoxically constrains $fr(\sim a)$ and hence $fr(a)$ to have a numerical value of ·5 when in fact $fr(a)$ is a free parameter. Unless we abandon the interchangeability of analytically equivalent statements in probability contexts, this argument cannot be faulted. Principle (1) is simply not acceptable *with unrestricted substitutability for its free variables.* It still remains to see, however, whether the needed exclusions are specific to the content of (1) or are merely technical qualifications binding on all principles of propositional probability, akin to forbidding division of zero by zero in otherwise universal theorems of algebra.

For example, one technically essential but conceptually trivial boundary restriction on (1) is evident in the following argument: Let u be the property of self-identity—i.e. for any entity x, $u(x)$ iff $x=x$—and let r and s be any two numbers different both from each other and from unity. Since it is logically true that $fr(u)=1$, E^{u}_{r} and E^{u}_{s} are both logically false and hence logically equivalent to each other. (In fact, if r and s are any entities other than numbers in the unit interval, this will be true regardless of what property u may be.) Consequently, from (1)

$$r=P(A, \ E^{u}_{r})=P(A, \ E^{u}_{s})=s,$$

contrary to stipulation that $r\neq s$. The source of this absurdity is allowing (1) to subsume instances in which E^{a}_{r} is logically false. That trouble should arise from this is not, however, at all peculiar to (1). For under any standard axiomatisation of propositional probability, it is elementary to show that for any two propositions A and B such that B entails A, $P(A, B)=1$ and $P(\sim A, B)=0$ so long as the quantity $P(A, B)$ is well-defined. Hence if $P(A, F)$ were to be well-defined for some logically false proposition F and another proposition A, it would inconsistently follow, since F then entails both A and $\sim A$, that $1=P(\sim A, F)=0$. Consequently, any consistent theory of propositional probabilities must treat as ill-defined all conditional probabilities in which the conditional's hypothesis is logically false. No matter how universally (1) may otherwise be conjectured to hold, therefore, we must in any event stipulate that it presupposes a logically consistent protasis E^{a}_{r}. Henceforth, I shall refer to this as the 'consistency requirement'.

At first impression, Miller's paradox seems to be a violation of the consistency requirement; for the argument involves hypothesising that $fr(a)=fr(\sim a)$ when we have also stipulated that $fr(a)$ has some value other than ·5. But while the assertion '$fr(a)=fr(\sim a)$ & $fr(a)\neq\cdot 5$' is indeed inconsistent, this conjunction does not figure in the formal deduction. Unwanted conclusion (2) follows merely by applying (1) to $E^{a}_{fr(\sim a)}$ and $E^{a}_{\cdot 5}$ separately, and each of these protases is in itself logically consistent (assuming a to be an empirical property) even if in fact $fr(a)$ has some value other than $\cdot 5$. To avert (2) we thus require a stronger restriction on (1) than just the consistency requirement.

To appreciate what this further restriction must be, consider first of all the fallacy in the following argument:

If Oswald's hand had trembled that moment in Dallas,
J. F. Kennedy would have been the U. S. President in 1964.
The U.S. President in 1964 was L. B. Johnson

[Therefore], if Oswald's hand had trembled that moment in Dallas,
J. F. Kennedy would have been L. B. Johnson.

Syntactically, this inference may be parsed according to the schema

$$\begin{array}{c} F(x) \\ \underline{x=y} \\ F(y) \end{array}$$

which is formally valid. However, *formally* valid inference schemata, when instantiated, yield *logically* valid inferences only when each free syntactic variable is replaced by a term which maintains the same referent throughout the argument.[1] This rule is deliberately violated in the present case by construing the descriptor 'the U.S. President' to have a context-dependent referent which varies from one premise to another. While considerably more could be said about this example, it suffices to demonstrate that formally valid arguments in which hypothetical premises are involved may well prove logically treacherous if these include descriptors.

Once one stops to think about it, it is evident that Miller's paradox results from precisely this sort of referential ambivalence. For insomuch as $\vdash E^{a}_{fr(\sim a)} \equiv E^{a}_{\cdot 5}$, the statement

(3) $$P(A, E^{a}_{fr(\sim a)})=fr(\sim a)$$

is logically equivalent to

(4) $$P(A, E^{a}_{\cdot 5})=fr(\sim a),$$

which is hence a consequence of (1) iff (3) is. Yet (4) is not at all the sort of conclusion which principle (1) is intended to authorise. What has gone wrong here is that the rightmost occurrence of '$fr(\sim a)$' in (3) and (4) is construed to designate whatever number is *in fact* the value of $fr(\sim a)$ (or, more technically, whatever value is stipulated for this parameter elsewhere in the argument), with the result that (3) asserts that the probability of A, given that $fr(a)=fr(\sim a)$, equals the actual frequency of $\sim a$. But what (1) says is that the probability of A, given that $fr(a)$ equals some particular number, has the same numerical value as the one stipulated for $fr(a)$ by the hypothesis. Thus while (3) is formally an instance of (1), its descriptor ambivalence disqualifies it as a *logical* consequence of the latter.

[1] For discussion of the distinction between formal validity and logical validity, see Rozeboom, 1962, p. 17.

It may be noted, incidentally, that the equivalence of '$fr(\sim a)=\cdot 5$' with

(5) $$fr(a)=fr(\sim a)$$

requires that (5) be interpreted as

(5′) *Hypothesis:* $fr(a)$ and $fr(\sim a)$ have the same value.

But while (5′) is the most natural reading of (5), a perfectly legitimate alternative interpretation is

(5″) *Hypothesis:* $fr(a)$ equals that number which is *in fact* the value of $fr(\sim a)$.

That (5′) and (5″) are very different hypotheses may be appreciated by reflecting that (5′) entails that $fr(\sim a)=\cdot 5$ whereas (5″) does not. Rather, combined with the auxiliary hypothesis, say, that $fr(\sim a)$ in fact equals $\cdot 2$, (5″) becomes the hypothesis that $fr(a)=\cdot 2$.

Without attempting to do full justice to the complexities of the situation here revealed, we may say that each occurrence of a descriptor generally carries a rider—usually implicit in the context of usage—resolving uncertainty about the descriptor's intended referent. In most indicative contexts, a descriptor d designates whatever entity in fact satisfies d. In subjunctive or counterfactual contexts, however, d's referent is often construed to be whatever it *would* be were some state of affairs s to be the case, or what it would have to be in order for a certain sentence $S(d)$ incorporating d to be true. Consequently, two different occurrences of the same descriptor in a hypothetical argument cannot be
same treated *logically* as the term, despite their formal (i.e. syntactic) identity, unless they carry the same reference rider. In particular, formally valid deductions from any conditional-probability generalisation may well eventuate in logical fallacy unless the restriction is imposed that no instantiation of a free (or universally quantified) variable is allowed by an expression whose referent is context dependent. We may call this the 'univalence requirement'. In no way does the univalence requirement diminish a generalisation's universality; it merely forbids certain syntactic manoeuvres which are valid formally but not logically.

2

So long as descriptor '$fr(\sim a)$' is allowed to have different reference riders in different contexts of occurrence, (3) cannot be deduced from (1) without violating the univalence requirement; hence this restriction thwarts Miller's paradox. (To be sure, (1) still entails (3) when the latter is read as 'The probability of A, given that $fr(a)=fr(\sim a)$, equals the value that $fr(\sim a)$ would have if $fr(a)=fr(\sim a)$'; but then conclusion (2) correspondingly reads '$\cdot 5$ is the value that $fr(\sim a)$ would have if $fr(a)=fr(\sim a)$', which is hardly paradoxical.) However, it still remains to see whether the restrictions we have found necessary to impose on (1) so far also cut the ground from under Popper's two variants of the paradox. One of Popper's versions (1966, p. 65) is essentially the same as Miller's and is similarly defeated by the univalence requirement. Popper's main version (1966, p. 64), however, cannot be disposed of quite this briskly. Essentially in Popper's own words, the argument goes as follows:

For any x whatever,

(6) If $\vdash s=2r-x$, then the following equivalence holds analytically:

$$x=r \text{ if and only if } x=s.$$

By substituting '$fr(a)$' for 'x' we obtain from (6)

(7) If $\vdash s=2r-fr(a)$, then the following equivalence holds analytically:
$$fr(a)=r \text{ if and only if } fr(a)=s.$$

By using the metalinguistic names 'E^a_r' and 'E^a_s', this can be written

(8) If $\vdash s=2r-fr(a)$, then $\vdash E^a_r \equiv E^a_s$.

But (8) allows us to substitute 'E^a_r' and 'E^a_s' in every probability formula for each other. Thus from (1) we obtain

(9) If $\vdash s=2r-fr(a)$, then $P(A, E^a_r)=P(A, E^a_s)=r=s$

and therefore

(10) If $\vdash s=2r-fr(a)$, then $r=s$.

But this result, it can be seen at once, is inconsistent; for we may for any given number, $fr(a)$, choose an r and an s such that r is not equal to s even though the conditions of (10) are satisfied.

At no point does the wording of this argument manifestly violate the consistency or univalence restriction on (1). It does, however, subtly equivocate between two different readings of the symbol '$fr(a)$', and when this ambiguity is resolved the argument fails in one of three ways. Observe to begin with that step (7), which on first impression seems to assert that certain statements are logically equivalent if a certain other statement is logically true, is in fact only the *schema* of such an assertion and becomes a genuine proposition only when the metalinguistic variables 's', 'r', and '$fr(a)$' are replaced by actual expressions in the object-language.[1] When this schema is instantiated, '$fr(a)$' can be replaced either by (*i*) a *numeral*, or by (*ii*) a descriptive phrase synonymous with 'the frequency of a'. If (*i*), the transition from step (7) to step (8) is invalid. But if (*ii*), then either the antecedent of the step (7) conditional is false or the consistency restriction on (1) blocks passage from (8) to (9), so that again no paradox is obtained. A specific example of each alternative will show how this is so:

For case (*i*), let 's', 'r' and '$fr(a)$' be instantiated by '$\cdot 8$', '$\cdot 4$', and '$\cdot 6$', respectively. This converts schema (7) into the determinate theorem

(7′) If $\vdash \cdot 8 = 2\text{x}\cdot 6 - \cdot 4$, then the following equivalence holds analytically:
$$\cdot 4 = \cdot 6 \text{ if and only if } \cdot 4 = \cdot 8$$

Observe however, that the apodosis of this conditional, namely, that $\vdash \cdot 4 = \cdot 6 \equiv \cdot 4 = \cdot 8$, is *not* equivalent to the assertion that $\vdash E^a_{\cdot 6} \equiv E^a_{\cdot 8}$; for the latter abbreviates $\vdash fr(a) = \cdot 6 \equiv fr(a) = \cdot 8$, in which '$fr(a)$' is not a metalinguistic variable but a synonym for the object-language descriptor 'the frequency of a'. The antecedent and hence also the consequent of conditional (7′) are true but do *not* entail the consequent of the corresponding instantiation of (8), namely, that 'the frequency of a is $\cdot 6$' is analytically equivalent to 'the frequency of a is $\cdot 8$'.

Alternatively—case (*ii*)—let 's' and 'r' be respectively instantiated by '$\cdot 8$' and '$\cdot 6$' as before, but this time let '$fr(a)$' be read as a synonym for object-

[1] Or, more precisely, by metalinguistic designators of such expressions. A certain amount of harmless waffling between use and mention is inescapable here unless we resort to tediously complicated formulations.

language descriptor 'the frequency of a'. Schema (7) then becomes the determinate theorem

(7″) If $\vdash .8=2\text{x}.6-fr(a)$, then the following equivalence holds analytically:

$$fr(a)=.6 \text{ if and only if } fr(a)=.8.$$

This is indeed equivalent to

(8″) If $\vdash .8=2\text{x}.6-fr(a)$, then $\vdash E^a_{.6}\equiv E^a_{.8}$,

from which together with (1) we may conclude, so long as neither $E^a_{.6}$ nor $E^a_{.8}$ are logically false,

(9″) If $\vdash .8=2\text{x}.6-fr(a)$, then $P(A, E^a_{.6})=P(A, E^a_{.8})=.6=.8$.

The paradoxical conclusion that $.6=.8$, however, is conditional upon its being the case that $\vdash .8=2\text{x}.6-fr(a)$, i.e. upon its being logically true that $fr(a)=.4$. But if the latter is the case, then both $E^a_{.6}$ and $E^a_{.8}$ are logically false, whence (9″) no longer follows from (8″) and (1) under the consistency restriction on (1).

There are still other ways in which schema (7) might be instantiated, notably by letting 's' and 'r' be descriptors, but in all cases derivation of the paradox is thwarted either by counterfactuality of the resultant conditional or, if the instantiated $\vdash s=2r-fr(a)$ is in fact logically true, by the inconsistency of E^a_r or E^a_s. We may conclude, then, that unlike Miller's simple but effective own version of his paradox, Popper's elaboration is merely a demonstration of subtly invalid inference.

3

Having defended the substantive generality of (1) against previous objections, I shall now act turncoat—reluctantly, since (1) still has considerable intuitive appeal to me—and raise some qualms about its specific content.

Principle (1) is one possible answer to the still unsolved question of how propositional probabilities relate to statistical frequencies. Another is proposed by what, for want of a better name, may be called

> *The Thesis of Statistical Reduction:* If sentence '$a(x)$' asserts that entity x has property a, where term 'x' has no special analytic connection with predicate 'a' (see fn. 1), then the unconditional probability of $a(x)$ is $P[a(x)]=fr(a)$.

(There are actually as many versions of this *Thesis* as there are interpretations for the functor 'fr', notably, frequency, statistical probability, or observed frequency. We assume that 'fr' is to be given the same reading here as in (1), but otherwise its specific interpretation may be left open.)

Although the Thesis of Statistical Reduction seems intuitively more dubious than does principle (1), a slight modification of Miller's paradox shows that (1) actually entails it. For since '$a(x)$' may be taken as the 'A' in (1), substituting '$fr(a)$' for 'r' in (1) yields

(11) $$P[a(x),\ E^{a}_{fr(a)}]=fr(a).$$

But '$E^{a}_{fr(a)}$' simply abbreviates '$fr(a)=fr(a)$', which is tautologous. Insomuch as the conditional probability of any proposition given a tautology equals its unconditional probability, (11) is thus equivalent to

(11′) $$P[a(x)]=fr(a).$$

Hence (1) entails (11′).

Unhappily, however, the Thesis of Statistical Reduction suffers from a fatal flaw when asserted in full generality. For let sentence '$b(x,y)$' assert that entity x stands in relation b to entity y—e.g. 'John loves Mary', or 'Peter beats Michael the first time they play chess together'—and let 'b_{x*}' and 'b_{*y}' abbreviate the monadic predicates '$b(x,—)$' and '$b(—,y)$' respectively. Since by definition,

(12) $$\vdash b_{x*}(y) \equiv b(x,y) \equiv b_{*y}(x),$$

while from (11′),

(13a) $$P[b_{x*}(y)]=fr(b_{x*})$$
(13b) $$P[b(x,y)]=fr(b)$$
(13c) $$P[b_{*y}(x)]=fr(b_{*y}),$$

it follows by the principle that logically equivalent propositions have equal unconditional probabilities that

(14) $$fr(b_{x*})=fr(b)=fr(b_{*y}).$$

But it is just not true that (14) is generally the case. The frequency of persons loved by John, for example, has no analytic nor even nomic bearing on the frequency of persons who love Mary; while if Peter and Michael are both good chess players, the frequency of 'Peter beats —— the first time they play chess together' is high, the frequency of '—— beats Michael [etc.]' is low, and the frequency of '—— beats —— [etc.]' is ·5 less the frequency of draws.

If principle (1) is to be retained at all, therefore, it must be curtailed even more severely than is accomplished by the consistency and univalence requirements.[1] Precisely what restriction is most germane is not altogether evident.

[1] Or does perhaps the univalence requirement forbid inference of (11) from (1)? Since '$E^{a}_{fr(a)}$' is tautologous no matter what reference rider is attached to '$fr(a)$' therein (so long as it is the same for both occurrences), it would seem that we can construe this rider to be the same as the one holding for the rightmost occurrence of '$fr(a)$' in (11), thereby satisfying the univalence requirement. On the other hand, it might conversely be argued that occurrence of a descriptor in the protasis of *any* conditional gives it a somewhat different meaning from its occurrence outside of that hypothesis, so that (11) is not a permissible instance of (1). The fact that this point is not entirely clear shows that my previous discussion of the univalence requirement is incomplete, and that the functioning of descriptors in hypothetical inference still lacks a fully satisfactory analysis. (*Retraction added in proof:* It now seems clear to me that the univocality requirement does indeed forbid inference of (11) from (1), or more precisely, that it authorizes the move to (11) only when $E^{a}_{fr}(a)$ is construed in a sense which is not tautological. Oh, well; the argument from (1) to (11′) is intriguing even if invalid, and its failure does no harm to the remaining argument.)

Our primary intuitive understanding of (1), however, is that its open variable 'r' is to be replaced by a specific numeral or at least by an expression whose referent can be identified by us. (Instances of (1) just don't *feel* quite right when the r-term designates an unknown, e.g. 'the proportion of 23rd Century astronauts with blonde hair'.) As a working hypothesis, therefore, let us provisionally impose on (1) the 'determinateness' requirement that its universality is to cover only numerical instances, i.e. that a conditional probability assertion of form (1) is authorised by this principle only when 'r' is, or is analytically equivalent to, a specific numeral. Under this determinateness requirement, (11) is no longer an authorised instance of (1), so paradox (14) is thereby averted. (Note, however, that this does not salvage the Thesis of Statistical Reduction; it merely prevents (1) from entailing it. What might be done to resuscitate principle (11′) will not be discussed here.)

Even so, the considerations underlying paradox (14) continue to haunt principle (1) no matter how severely the latter's generality is restricted. Let b_{x*} and b_{*y} be defined from binary relation b as before, so that (12) continues to hold. Then what is the conditional probability of $b(x,y)$ given both that $fr(b_{x*}) = r$ and that $fr(b_{*y}) = s$, where 'r' and 's' are specific numerals such that $r \neq s$? For example, what is the probability that Peter beats Michael the first time they play chess together, given that Peter wins 60 per cent of his first chess games with other players while Michael is beaten on 10 per cent of his first chess games with others? By principle (1), the probability is ·60 that Peter beats Michael given only that Peter wins 60 per cent of his games, whereas the probability is ·10 that Peter beats Michael given only that Michael is beaten on 10 per cent of his games. There is no inconsistency in these probabilities, for each is conditional on a different statistical hypothesis. But these two statistics are entirely compatible, and if they are given jointly, *then* what is the conditional probability that Peter beats Michael? The answer—or rather, non-answer—is that intuition simply breaks down in this case. To be sure, we can think of possibilities, such as

$$P[b(x,y),\ E_r^{b_{x*}}\ \&\ E_s^{b_{*y}}] = {\cdot}5(r+s), \tag{15}$$

but (15) is in general false (cf. the case where r or s is zero or unity) and there are many alternatives to (15) which might be entertained, each as intuitively arbitrary as the others.

Insomuch as there exists no general principle which, for any three propositions A, B, and C, derives $P(A, B\ \&\ C)$ from $P(A, B)$ and $P(A, C)$, there is no reason why (1) should be held responsible for providing the numerical value of $P[b(x,y), E_r^{b_{x*}}\ \&\ E_s^{b_{*y}}]$. The plausibility of (1), however, is now seen to depend on the strength of one's conviction that there exists a numerical function f such that for any two numericals r and s,

$$P[b(x,y),\ E_r^{b_{x*}}\ \&\ E_s^{b_{*y}}] = f(r, s). \tag{16}$$

For if there is no such f, i.e. if the specific numerical frequencies of $b(x,—)$ and $b(—, y)$ do not suffice to determine the conditional probability of $b(x,y)$ given these frequencies, then whatever additional factors influence this probability may well be expected to make a difference for $P[a(x), E_r^a]$ as well, thereby preventing the latter from being determined solely by frequency datum E_r^a as alleged by (1). Personally, I consider it fairly plausible that, under suitable boundary restrictions, generalisation (16) is in fact true for some *f*. But until this function has actually been identified and submitted to probate, (16) and hence (1) must remain an article of more or less tenuous faith.

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