First published in *Synthese*, 1966, *16*, 170-233.

Scaling Theory and the Nature of Measurement

Among the foundational issues of scientific methodology, the theory of measurement enjoys a notable distinction—namely, *attention*. For while most conceptual procedures in technical science still remain appallingly underexamined by serious metascience, a voluminous literature has formed around the topics of measurement and scaling, especially in the behavioral sciences of the past two decades. Moreover, the swelling chorus of these contributions has achieved a harmony which increasingly approaches unison, its composition being Campbell's (1920) classic theme embellished by modern set-theoretical notions of a formal representation system (e.g., Suppes & Zinnes, 1963), together with secondary motifs from Stevens' (1951) theory of scale types and a still unfinished coda on 'conjoint measurement' (Luce & Tukey, 1964).

It is no intent of mine to suggest that anything is basically amiss with this development. Quite the opposite: What has happened in measurement theory is convincing evidence that powerful advances in scientific metatheory *are* possible when rigorous thinkers are willing to put some intellectual muscle into the enterprise. Even so, the tonal balance of current measurement theory does not ring true to my ear. Importantly distinct melodic lines have become fused where they should be played at counterpoint. The theory of scale types has effected a strange inversion of the 'meaningfulness' air which I find teeth-grittingly discordant. And certain fundamental tones which I would score forte are at present scarcely audible. What I shall here attempt, therefore, is a new orchestration of this material. At no one place will my version differ radically from standard doctrine on these matters; but through an accretion of differences in emphasis and phrasing I hope to convey a perspective on measurement theory which has greater breadth, solidity, and extrapolative thrust than has been attained previously.

Actually, my primary concern will be with *scaling*, not measurement. For I shall argue that 'measurement' in the tough sense of the word must be distinguished from scaling, and that very little of the literature on 'measurement theory' has had anything to say about genuine measurement at all. Even so, scaling and measurement are so intimately connected that any account of the one remains seriously incomplete until it examines the other as well. Also closely linked to these, but nonetheless importantly distinct from them despite the confounding which has burgeoned in the recent measurement-theory literature, are factorial decomposition methods of data analysis.

The organization of this essay, then, will be as follows: First we shall formulate in considerable generality the definitions of ‘scale’, ‘scale type’, and ‘scale interpretation’, the last being the pivotal concept of scaling theory. Next, we undertake an abstract characterization of factorial decomposition, which differs *formally* from scaling only in a few small technical details which, however, reflect a vital methodological distinction. And finally, measurement proper will emerge as but a species under the genus of scale interpretation, albeit by far and away the most important one. That the discussion will become increasingly programmatic and speculative in its latter stages is an unfortunate necessity for which I ask the reader's forbearance. The methodological vistas encountered there are so new that I am able to chart but a few rough compass bearings in terrain which still awaits its pioneers.

1 Scales

Scaling theory is remarkable for the fact that its cognitive content consists almost entirely of stipulative definitions—no postulates are required, and its major theorems are deductively trivial. To grasp the full significance of these abstractions, however, is apparently no small thing. For the aura of mysterious profundity which has come to invest so much of the scaling literature is sustained largely by a dearth of clear thinking about the matters at issue.

The essential technical concepts upon which scaling theory rests are ‘property’, ‘relation’, ‘function’, and ‘(scientific) variable’. Though all but the last of these are familiar notions in logic and mathematics, their review will make an appropriate beginning if only to introduce present terminology and notation.

Background Concepts

With considerable reluctance, I shall here provisionally adopt an extensional definition of ‘property’ (or, what is essentially the same, ‘attribute’), namely, that properties (attributes) are classes of the entities whose properties they are. That is, we shall suppose a property P over a domain $\mathbf{d}$ to be a subset $\mathbf{p}$ of $\mathbf{d}$ such that a member d of $\mathbf{d}$ has property P iff[1] d belongs to $\mathbf{p}$. For example, human ‘baldness’ (or ‘being bald’) is to be identified with the class of all bald humans, while over the domain comprising all chunks of minerals, the property ‘crystalline’ is the class of all crystalline rocks. What is objectionable about this usage, of course, is that ‘properties’ are really distinguishing features of the entities which possess them, so that in principle, properties can be coextensive even though non-identical.[2] (Thus

[1] As the reader is doubtlessly aware, ‘iff’ is a standard abbreviation for ‘if and only if’.

[2] The difference between properties and classes is often characterized as the difference between a predicate's *intension* and *extension*, the latter being the class of things which satisfy it while the former is held to be the predicate's meaning. However, I have argued elsewhere (Rozeboom, 1962a) that properties are what predicates designate, and must hence be distinguished from both

if all crystalline rocks were translucent and conversely, we should still deny that crystallinity and translucency are the same property of rocks even though the class of crystalline rocks would be identical with the class of translucent rocks.) However, equating properties with classes has become standard practice in contemporary set-theoretical approaches to scaling, while in this context any technically explicit attempt to preserve the distinction uselessly complicates the analysis at most points, especially since our intuitive ability to discriminate between coextensive properties fails altogether for abstract domains. (E.g., are 'being even' and 'being one greater than an odd number' the same or different attributes of positive integers?) Hence an extensional introduction to scaling theory can be justified pragmatically, especially insomuch as once the extensional version is well understood, the modifications required to do justice to the class/property distinction are minor and obvious. Even so, since the most challenging problems of scaling and measurement cannot be expressed at all in extensional terms, I shall differentiate notationally between (a) sets (classes) which are here truly sets and (b) properties whose present set-characterization is only provisional by using boldface symbols for the former and italics for the latter. Similarly, as much as possible I shall write predications in logistical rather than class-membership notation.

A *relational property*, or simply 'relation', is a property over a domain of ordered k-tuples of entities, where k is some integer appropriate to the relation in question and the latter is said, more specifically, to be a 'k-adic' or 'k-ary' relation. That is, if $\mathbf{e}_1, \ldots, \mathbf{e}_k$ are k not-necessarily-distinct sets of entities, then any property over the product set $\mathbf{e}_1 \times \mathbf{e}_2 \times \cdots \times \mathbf{e}_k$ is a k-adic relation.[3]

For example, the binary relation 'x is the husband of y', construed extensionally, is the set of all ordered pairs of persons such that the first person is a man, the second is a woman, and the first is married to the second; while the ternary numerical relation 'lies between' is the set of all triples of numbers such that the first is larger than one of the latter two and is smaller than the other. In logistical notation, the formula '$R^k(e_1, \ldots, e_k)$' asserts that entities $e_1, \ldots, e_k$ stand in relation R^k; whereas in set-theoretical notation this same idea is expressed '$\langle e_1, \ldots, e_k \rangle \in \mathbf{r}^{(k)}$', where $\mathbf{r}^{(k)}$ is some subset of a k-fold product domain. When all the k-tuples which belong to relation $\mathbf{r}^{(k)}$ are members of the k-fold product of a set $\mathbf{e}$ with itself—i.e., if $\mathbf{r}^{(k)} \subset \mathbf{e}^k$—we shall say simply that $\mathbf{r}^{(k)}$ is a k-ary relation over $\mathbf{e}$.

A *function* over a set $\mathbf{a}$ into a set $\mathbf{v}$ may now be defined as any subset ϕ of product set $\mathbf{a} \times \mathbf{v}$ which has the special feature that for each element a in $\mathbf{a}$, there

---

the predicate's extension and its meaning.

[3]The reader whose knowledge of set-theoretical terminology is scanty may like to be reminded that a 'product set' $\mathbf{e}_1 \times \mathbf{e}_2 \times \cdots \times \mathbf{e}_k$ is the set of all ordered k-tuples such that a k-tuple $\langle e_1, \ldots, e_k \rangle$ belongs to $\mathbf{e}_1 \times \cdots \times \mathbf{e}_k$ iff $e_1 \in \mathbf{e}_1, \ldots, e_k \in \mathbf{e}_k$. When $\mathbf{e}_1, \ldots, \mathbf{e}_k$ are all the same set $\mathbf{e}$, product set $\mathbf{e}_1 \times \cdots \times \mathbf{e}_k$ may be written more compactly as $\mathbf{e}^k$.

exists one and only one element v of $\mathbf{v}$ such that $\langle a, v\rangle \in \phi$. The elements of $\mathbf{a}$ are known as the *arguments* of function ϕ, while if a is an argument of ϕ, the element v of $\mathbf{v}$ for which $\langle a, v\rangle \in \phi$ is the *value* of ϕ for a. Correspondingly, we may call sets $\mathbf{a}$ and $\mathbf{v}$ the *argument domain* and *value domain*, respectively, of function ϕ, while the set $\mathbf{v}'$ of those elements of $\mathbf{v}$ which are actually a value of ϕ for some argument is the *range* of ϕ.

We shall adopt the compound symbol 'ϕa', in which the function symbol is prefixed to an argument term 'a' without parentheses around the latter, to designate the value of ϕ for a.

A function ϕ from argument domain $\mathbf{a}$ into value domain $\mathbf{v}$ also defines a mapping of properties, relations, and other constructions on $\mathbf{a}$ into corresponding properties, etc., on $\mathbf{v}$. This idea may be made notationally explicit as follows: If a^k is a k-tuple of $\mathbf{a}$-elements (i.e., $a^k \in \mathbf{a}^k$), then '$\phi^* a^k$' designates the k-tuple in $\mathbf{v}^k$ whose components are the values of ϕ for the corresponding components of a^k, i.e.,

DEF. 1. $$\phi^*\langle a_1, \ldots, a_k\rangle =_{\text{def}} \langle \phi a_1, \ldots, \phi a_k\rangle;$$

while if $\mathbf{r}^{(k)}$ is a k-ary relation over $\mathbf{a}$ (i.e., $\mathbf{r}^{(k)} \subset \mathbf{a}^k$), then $\phi^*\mathbf{r}^{(k)}$ is the set of k-tuples in $\mathbf{v}^k$ into which ϕ maps the k-tuples in $\mathbf{r}^{(k)}$—i.e.,

DEF. 2. $\phi^*\mathbf{r}^{(k)} =_{\text{def}}$ The set which contains a given k-tuple v^k iff v^k is the value of ϕ^* for some k-tuple in $\mathbf{r}^{(k)}$,

or expressed logistically,

DEF. 2a. $$\phi^* R^k(x^k) =_{\text{def}} (\exists a^k)[R^k(a^k) \cdot x^k = \phi^* a^k]$$

where R^k and $\phi^* R^k$ are k-adic relations over the argument and value domain, respectively, of ϕ. The symbol 'ϕ^*' in itself may be construed to designate the function whose argument domain $\mathbf{a}^*$ is the set of all subsets of all product sets of ϕ's argument domain $\mathbf{a}$, whose range $\mathbf{v}'^*$ is the set of all subsets of all product sets of ϕ's range $\mathbf{v}'$, and whose value for a given argument in $\mathbf{a}^*$ is defined by Def. 2.[4] We may call extended domains $\mathbf{a}^*$, $\mathbf{v}^*$, and function ϕ^* the 'first-level developments' of $\mathbf{a}$, $\mathbf{v}$, and ϕ, respectively.

[4]This implies that when I write '$v^k = \phi^* a^k$' (cf. Def. 1) for k-tuples v^k in $\mathbf{v}^*$ and a^k in $\mathbf{a}^k$ what I really meant is that ϕ^* maps the unit class of a^k into the unit class of v^k. However, no harm will be done, and much notational simplification achieved, if the same symbol is used ambiguously to designate both an entity and its unit class. Also for simplicity, I shall write '$\phi^*\mathbf{R}$' for the set of relations into which ϕ^* maps the elements in a set $\mathbf{R}$ of relations over ϕ's arguments. (Strictly speaking, the proper notation would be '$\phi^{**}\mathbf{R}$'.)

The $\mathbf{v}$-relation $\phi^*\mathbf{r}^{(k)}$ into which a function ϕ from $\mathbf{a}$ into $\mathbf{v}$ maps a k-ary relation $\mathbf{r}^{(k)}$ over $\mathbf{a}$ may be referred to as *the image in* $\mathbf{v}$ *of* $\mathbf{r}^{(k)}$ *under* ϕ, or 'the ϕ-image of $\mathbf{r}^{(k)}$' for short. Conversely, if $\mathbf{q}^{(k)}$ is a relation over ϕ's range, the set of all k-tuples in $\mathbf{a}^k$ whose ϕ-images are in $\mathbf{q}^{(k)}$ may be called *the contraimage in* $\mathbf{a}$ *of* $\mathbf{q}^{(k)}$ *under* ϕ (for short, 'the ϕ-contraimage of $\mathbf{q}^{(k)}$') and symbolized '$\phi^I\mathbf{q}^{(k)}$'). That is, expressed logistically,

Def. 3 $$\phi^I Q^k(x^k) =_{\text{def}} Q^k(\phi^* x^k).$$

The symbol 'ϕ^I' itself may be thought to name the function—call it ϕ's 'first-level contra-development'—whose argument domain is the first-level development of ϕ's range, whose value domain is the first-level development of ϕ's argument domain, and whose value for any given argument is the latter's ϕ-contraimage. An immediate consequence of ϕ^I's definition is that

Theorem 1. For any relation $\mathbf{q}^{(k)}$ over the range of a function ϕ, and any k-tuple a^k of ϕ's arguments, the ϕ-image of a^k belongs to $\mathbf{q}^{(k)}$ iff a^k belongs to the ϕ-contraimage of $\mathbf{q}^{(k)}$.

A function is said to be *one-one* if it has a different value for every different argument, and is *many-one* otherwise. If ϕ is a one-one function from $\mathbf{a}$ into $\mathbf{v}$, there exists a function from the range $\mathbf{v}'$ of ϕ into $\mathbf{a}$, designated 'ϕ^{-1}' called the *inverse* of ϕ, such that the value of ϕ^{-1} for an argument v in $\mathbf{v}'$ is the element of $\mathbf{a}$ whose value of ϕ is v. It is easily seen that whenever ϕ^{-1} exists, an argument a of ϕ is the value of ϕ^{-1} for an element v of ϕ's range iff a is the ϕ-contraimage of v, and hence, more generally, that $\phi^{-1*} = \phi^I$. Moreover, since $v = \phi a$ and $a' = \phi^{-1}v$ jointly entail $a' = a$,

Theorem 2. If ϕ is a one-one function, then any relation $\mathbf{r}^{(k)}$ over ϕ's argument domain is the ϕ-contraimage of its ϕ-image—i.e., for any relations $\mathbf{r}^{(k)}$ and $\mathbf{q}^{(k)}$, $\mathbf{r}^{(k)} = \phi^{-1}\mathbf{q}^{(k)}$ iff $\mathbf{q}^{(k)} = \phi^*\mathbf{r}^{(k)}$. *Corollary*: If ϕ is a one-one function, then for any relation $\mathbf{r}^{(k)}$ and any k-tuple a^k of ϕ's arguments, a^k belongs to $\mathbf{r}^{(k)}$ iff the ϕ-image of a^k belongs to the ϕ-image of $\mathbf{r}^{(k)}$.

If ϕ is *not* one-one, however, the biconditionals in Theorem 2 and its corollary are weakened to conditionals.

Finally, we note that if ϕ_2 is a function from set $\mathbf{m}$ into set $\mathbf{v}$ while ϕ_1 is a function from set $\mathbf{a}$ into a subset of $\mathbf{m}$, then there exists a function from $\mathbf{a}$ into $\mathbf{v}$, designated '$\phi_2\phi_1$' and known as the *product* or *composition* of ϕ_2 with ϕ_1, whose value for an argument a is the element of $\mathbf{v}$ which is the value of ϕ_2 for the element of $\mathbf{m}$ which is the value of ϕ_1 for a. That is, $\phi_2\phi_1 a$ is the value of ϕ_2 for element $\phi_1 a$ of $\mathbf{m}$.

Scientific Variables

We are now ready to formalize the all-important notion of 'variable' in the scientific sense of this distressingly ambiguous word. The role of this concept in technical science is complex and needful of *much* more reconstructive analysis than it has received to date. I have elsewhere (Rozeboom, 1961) explored the formal properties of scientific variables at some length (though still far from exhaustively) and will here repeat only so much of the analysis as present purposes require.

In brief, the scientific conception of 'variable' brings technical efficiency to study of the properties of things. A field of scientific inquiry is delimited (albeit seldom very precisely) by a class of objects or events which may be called the science's 'subject domain', together with a more or less restricted set of properties and relations over this domain whose incidence and intercorrelations constitute the science's concern and which may be called its 'scope'. Although a large proportion of a science's most advanced technical labor is given to development of precise, reliable criteria for the properties included in its scope, practical problems of definition and detection are not now at issue. What is germane here is that the properties in a science's scope always come in clusters such that the properties within each cluster are mutually exclusive and jointly exhaustive over their subject domain. For example, each of the property sets indicated by the following predicate schemata is such that for each member of the domain of time-slices of organisms, one and only one property in the set is true of that organism at that time:

> belongs to species ____ [variously fill the blank with names of all different species];
>
> weighs ____ lbs. [variously fill the blank with nonnegative real numbers];
>
> ____ exposed to a 40 db white noise [variously fill the blank with 'is' or 'is not'].

Such property clusters are basically what scientists mean by the term 'variable', except that while this is the meat of the concept, its more advanced employments require a formalization slightly more sophisticated than this.

Accordingly, I shall distinguish between *natural*, or *unscaled*, variables on the one hand, and *formal*, or *scaled*, variables on the other. 'Natural' variables are what arise most directly out of a science's scope, namely, the property clusters just cited. Specifically,

Def. 4. If $\boldsymbol{\alpha}$ is a set of properties over a subject domain $\mathbf{d}$ such that for each member d of $\mathbf{d}$ there is one and only one α in $\boldsymbol{\alpha}$ such that $\alpha(d)$, then $\boldsymbol{\alpha}$ is a natural, or unscaled, variable over $\mathbf{d}$.

If $\boldsymbol{\alpha}$ is a natural variable over domain $\mathbf{d}$, then obviously there also exists a function from $\mathbf{d}$ into the set of all subsets of $\mathbf{d}$ whose value for a member d of $\mathbf{d}$ is the property in $\boldsymbol{\alpha}$ which is possessed by d. Although for most purposes we need not distinguish between this function and natural variable $\boldsymbol{\alpha}$ itself, the former will be symbolized by '$\dot{\boldsymbol{\alpha}}$' when technical precision is desired. I shall consistently use the phrase 'the value of (natural variable) $\boldsymbol{\alpha}$ for (its argument) d' to designate the value of function $\dot{\boldsymbol{\alpha}}$ for d, namely, the property α in $\boldsymbol{\alpha}$ such that $\alpha(d)$.

In contrast to natural variables, which are sets of properties over a science's subject domain, 'formal' variables are functions over the latter. Specifically,

Def. 5. If ϕ is a function from domain $\mathbf{d}$ into a set $\mathbf{n}$ of abstract entities, then ϕ is a formal, or $\mathbf{n}$-scaled, variable over $\mathbf{d}$.

I have obscurely characterized the values of a formal variable as 'abstract entities' in order to avoid premature restrictions as to their admissible nature. In practice, $\mathbf{n}$ is usually a set of real numbers, and for this reason I shall henceforth refer to the values of a formal variable as *nums* when their more specific nature is left open. That is, 'num' will be a systematically ambiguous term, like an unspecified parameter in an algebraic equation, which is to be heuristically understood as 'number', but which at times will receive interpretations other than this.

To appreciate the distinction between a natural (unscaled) variable and a formal (num-scaled) counterpart thereof, consider the predicate schema

weighs ____ lbs.

By alternatively filling the blank with the names of all positive real numbers, we generate a set of predicates which collectively designate a set of weight properties which are mutually exclusive and exhaustive over any subject domain of time-slices of things. This set of weights is the natural variable Weight, and if John Smith, today, weighs 163 lbs., the value of the unscaled *Weight* variable for Smith, today, is *weighs 163 lbs.* To describe the variable's value in this case we must employ the full predicate 'weighs 163 lbs.' or its logical equivalent (e.g., 'weighs 2,608 oz.'), *not* just the numeral '163' nor even '163 lbs.'. On the other hand, the predicate schema 'weighs ____ lbs.' may also be taken to define a formal variable, *Weight-in-lbs.*, whose value for an individual weighing n lbs. is the number n. The value of the formal Weight-in-lbs. variable for John Smith today is the *number* 163, not the property of weighing 163 lbs. nor even the 'denominate number' 163 lbs.

In the preceding example, the nums which are the formal variable's values are, in fact, numbers. That this is not necessarily the case is illustrated by the predicate schema has ____-colored hair.

When this is variously completed with alternative color descriptions it generates a natural variable over hirsute creatures whose value for an organism whose hair color is C is the attribute 'having C-colored hair'. But it may also be taken to define a function from hirsute creatures into colors such that the value of this function for any given argument is the color of the latter's hair. So construed, *Hair-color* becomes a num-scaled formal variable whose 'nums' are colors.

It should be noted that for any num-valued formal variable ϕ over domain $\mathbf{d}$, there exists a corresponding natural variable $\boldsymbol{\alpha}_\phi$ over $\mathbf{d}$ whose value for any d such that $\phi d = n$ is the property of having a ϕ-scale value equal to n. When 'properties' are interpreted extensionally, the difference between ϕ and $\boldsymbol{\alpha}_\phi$ is that the value of ϕ for any d in $\mathbf{d}$ is an element of num domain $\mathbf{n}$ (i.e., paradigmatically a *number*), whereas the value of $\boldsymbol{\alpha}_\phi$ for d is a subset of $\mathbf{d}$, namely, those d' such that $\phi d' = \phi d$.

Semantic Scales vs Formal Scales

Just as every formal variable has an unscaled (natural) counterpart, so is it the case that to every natural variable there corresponds not merely one but a multitude of num-valued formal variables, and conceptually replacing a natural variable with one of its formal equivalents is basically what is meant by 'scaling'. In relatively unsophisticated contexts, however, scaling is often a half step which presents the verbal trappings of a formal scale without actually adopting its substance. In such instances, symbols which ordinarily name nums are adapted to serve as code abbreviations for the predicates describing the values of a natural variable.

Suppose, for example, the administration of Brainsweat University classifies each resident student according to his major subject—e.g., Chemistry major, Psychology major, French major, etc., together with one or two wastebasket categories like 'unknown' or 'other' to deal with otherwise unclassifiable cases. This defines a natural *College-major* variable over the domain of Brainsweat U. students whose value for any student whose major subject is X is the property 'majoring in X'. However, insomuch as the predicate schema 'is majoring in ____' is hopelessly unwieldy for efficient processing of student records, the administration has assigned to each College-major alternative a code numeral—say '4' for Chemistry, '8' for Psychology, '13' for French, etc.—for recording college-major statuses in its files. Persons familiar with this code can then with perfect propriety make statements such as 'John Smith's major is 4', 'Mary Jones's major is 8', etc., without the slightest intimation that the college majors of these students involve *numbers* in any way; for these verbal expressions, when understood as intended, are merely shorthand for 'John Smith's major is Chemistry', 'Mary Jones's major is Psychology', etc. By the same token, it would be literally senseless to say 'Mary Jones's major is twice as large as John Smith's', or 'The average major at Brainsweat U. is 11.36', for these claims, upon translation, respectively contend that being a Psychology major is twice as large as being a Chemistry major, and that the sum of all the college-major statuses of Brainsweat U. students, divided by their number, is equal to a college-major attribute whose code numeral is '11.36'—which, cognitively, is complete gibberish, the former in that the linguistic expression 'Majoring in subject X is twice as large as majoring in subject Y' has no meaning, and the latter in that addition and division are not operations which are defined on college-majoring attributes, nor has any college major been assigned code numeral '11.36'.[5]

On the other hand, once a numerical code has been stipulated for the natural College-major variable, we can also derive from this a corresponding number-valued formal variable—call it *College-major score*—by defining the College-major score of a Brainsweat U. student to be that number which is normally designated by the numeral which, in the University's coding system, abbreviates that student's College-major category. Under this construction, the College-major scores of John Smith and Mary Jones are the *numbers* 4 and 8, respectively; and it is not only meaningful but impeccably correct to say that Mary Jones's College-major score is twice as large as John Smith's because the number 8 is, in fact, twice as large as the number 4. Similarly, asserting that the average College-major score at Brainsweat U. is 11.36 is perfectly sensible, albeit not necessarily accurate, because the average *number* into which the College-major-score function maps Brainsweat U. students may indeed be 11.36 even though there is no College-major attribute to which this number corresponds.

The two kinds of scaling just illustrated may be described generically as *semantic scaling* on the one hand, and *formal scaling* on the other.

DEF. 6. A scale-name A and set $\mathbf{a}$ of symbols compose a semantic scale for natural variable $\boldsymbol{\alpha}$ in a given language community iff, for every property α in $\boldsymbol{\alpha}$, there exists a unique element a of $\mathbf{a}$ such that within this language community the symbol sequence

$$\underline{\quad\quad}\text{'s value of } A \text{ is } a$$

(or some paraphrase thereof) means '____ has property α.'

Since in paradigm instances of semantic scaling the set $\mathbf{a}$ consists of numerals—

[5]Strictly speaking, this charge of meaninglessness must be qualified somewhat in that when properties are construed to be classes, size comparisons and arithmetic operations are, in fact, well defined for them by set theory.

i.e., the symbols normally used to name numbers—we may generically refer to the elements of $\mathbf{a}$ as 'numes', and presume that in their primary linguistic contexts, numes designate nums. To an outsider or novice at scaling practices, semantic scales thus appear to find something numish about the data so scaled, and tempt inferences from these scalings which would be perfectly legitimate were the numes in this context actually referring to nums, but which, precisely because these numes do *not* here refer to their accustomed nums, are bereft of any cognitive content.

However, given a semantic scale $\langle A, \mathbf{a} \rangle$ for natural variable $\boldsymbol{\alpha}$ over domain $\mathbf{d}$, we can always derive from this a formal scale ϕ of $\boldsymbol{\alpha}$ by letting ϕ be that function over $\mathbf{d}$ whose value for each argument d is the num ordinarily designated by the nume which is the A-code abbreviation for d's value of $\boldsymbol{\alpha}$. More generally, without concern for the formal scale's conceptual origin,

DEF. 7. A function ϕ from domain $\mathbf{d}$ into a num domain $\mathbf{n}$ is a formal scale for a natural variable $\boldsymbol{\alpha}$ over $\mathbf{d}$ iff $\boldsymbol{\alpha}_\phi = \boldsymbol{\alpha}$—i.e., if to each α in $\boldsymbol{\alpha}$ there corresponds a num n in $\mathbf{n}$ such that for any d in $\mathbf{d}$, $\phi d = n$ iff $\alpha(d)$.

It follows immediately that ϕ is a formal scale for $\boldsymbol{\alpha}$ if and only if ϕ has the same argument domain as $\boldsymbol{\alpha}$ and there exists a function f from $\boldsymbol{\alpha}$ into the range of ϕ such that for any argument d, ϕd is the num into which f maps a property α iff α is the value of $\boldsymbol{\alpha}$ for d. Such an f will be called a *scaling transformation* of $\boldsymbol{\alpha}$ into ϕ. A scaling transformation of $\boldsymbol{\alpha}$ into a formal scale ϕ for $\boldsymbol{\alpha}$ must obviously be one-one, since otherwise, contrary to definition, the value of ϕ for an argument d would not suffice to determine d's value of $\boldsymbol{\alpha}$. Hence a function f is a scaling transformation of $\boldsymbol{\alpha}$ into ϕ iff $\dot{\boldsymbol{\alpha}} = f^{-1}\phi$.

It is evident that if a function ϕ whose range is included in num domain $\mathbf{n}$ is a formal scale for natural variable $\boldsymbol{\alpha}$, while g is some one-one function from $\mathbf{n}$ into another num domain $\mathbf{n}'$ not necessarily distinct from $\mathbf{n}$, then the product function $g\phi$, whose range is included in $\mathbf{n}'$, is also a formal scale for $\boldsymbol{\alpha}$. Conversely, if ϕ_1 and ϕ_2 are both formal scales for $\boldsymbol{\alpha}$, there exists a one-one function g (namely, $g = f_2 f_1^{-1}$, where f_1 and f_2 scale $\boldsymbol{\alpha}$ as ϕ_1 and ϕ_2, respectively) from the range of ϕ_1 into the range of ϕ_2 such that $\phi_2 = g\phi_1$. Hence if ϕ_1 is a formal scale for natural variable $\boldsymbol{\alpha}$, another formal variable ϕ_2 is also a scale for $\boldsymbol{\alpha}$ iff there exists a one-one mapping of ϕ_1-values into ϕ_2-values. Such a function, which converts one formal scale for $\boldsymbol{\alpha}$ into another, may be called a *rescaling transformation* of the first into the second, while one formal scale is a 'rescaling' of another iff there exists a rescaling transformation of the one into the other.

It is of some technical interest to note that so long as we place no restrictions on the kinds of entities which can serve as the 'num' values of a formal variable, the function $\dot{\boldsymbol{\alpha}}$ which maps the arguments of natural variable $\boldsymbol{\alpha}$ into their $\boldsymbol{\alpha}$-values is itself a formal scale for $\boldsymbol{\alpha}$. While this is a special case highly remote from what is normally envisioned in scaling practice, it serves to underscore the unusual breadth of the concept of 'scale'.

Scale 'Types' and Scale 'Interpretations'

Since every formal variable is a formal scale and conversely, the only difference between these two notions is that 'formal variables' are numvalued functions attended to for their own sake, whereas 'formal scales' are these same functions considered in their relation to the natural variables which they scale. More specifically, insomuch as 'scaling' is simply a methodology for expeditious processing of natural variables, our primary interest in formal scales lies in what we can infer from the properties of scaled data about the natural features of these data. That is, we want to know what a proposition about formal scales signifies in terms of the natural variables to which these scales correspond. Translation of propositions about scaled variables into propositions about their natural counterparts may be described as *interpreting* the scales in question. Thus the pragmatics of scaling has three major divisions: (1) What formal scales can be effectively defined for a given natural variable $\boldsymbol{\alpha}$? (2) What interpretations can be made of the formal scales available for $\boldsymbol{\alpha}$? And finally, (3) given a choice among formal scales for $\boldsymbol{\alpha}$, which ones are interpretively most acceptable? Scaling theory can say little about the first of these without examining the detailed definitions of $\boldsymbol{\alpha}$'s values, but with a little care it can easily set aright the confusions and false doctrines which still abundantly hold forth on matters (2) and (3).

Any proposition about a given variable, scaled or unscaled, makes reference to (a) one or more specific values of the variable, and/or (b) the variable as a whole. While a theory of scale interpretation which includes the latter case is awkward to develop in abstract generality, the only instances of (b) which appear to have any scientific importance are statistical assertions which can be reduced to or arbitrarily well approximated by propositions of type (a). Consequently, we may consider the primary concern of scaling theory to be assertions about specific values of variables—i.e., propositions which can be expressed logistically in the form '$R^k(x_1, \ldots, x_k)$', where '$R^k(\ \ )$' is a k-place sentence schema and each x_i is a particular value of a natural or formal scientific variable. For example,

On the average, organisms ____ are longer lived than organisms ____ . [A dyadic schema whose blanks are to be filled with predicates of form 'belonging to species S'.]

John Smith ____ . [A monadic schema whose blank can be filled with any monadic predicate such as 'is 71 inches tall', 'has brown hair', etc.]

Any object ____ is twice as heavy as any two combined objects ____ and ____ , respectively. [A triadic schema whose blanks are to be filled with predicates of form ‘weighing w lbs.’.]

____ is equal to or greater than ____ . [A dyadic schema whose blanks are to be filled with numerals.]

The first three of these generate statements about the values of certain natural variables, whereas the last is about numbers and hence about the values of any number-scaled formal variable.

If ‘$R^k(\ )$’ is a k-place sentence schema which becomes a true or false assertion when completed by any k-tuple of terms respectively designating values of a natural variable $\boldsymbol{\alpha}$, it may be construed to name a k-adic relation R^k over the domain of $\boldsymbol{\alpha}$-values. Similarly, a k-place sentence schema ‘$Q^k(\ )$’ which makes a true or false statement when completed by any k-tuple of numes designating nums in domain $\mathbf{n}$ defines a k-adic relation Q^k over $\mathbf{n}$. If $\mathbf{n}$ is the range of values for some formal scale ϕ of $\boldsymbol{\alpha}$, the possibility thus arises that $\boldsymbol{\alpha}$-relation R^k may be so coordinated with num relation Q^k that information concerning Q^k for a k-tuple of ϕ-scale values reveals how the corresponding k-tuple of $\boldsymbol{\alpha}$-values stand in respect R^k.

More specifically, suppose that ϕ is a num-valued formal scale for natural variable $\boldsymbol{\alpha}$ derived from the latter by scaling transformation f—i.e., $\phi = f\dot{\boldsymbol{\alpha}}$. Then every k-tuple α^k of properties in $\boldsymbol{\alpha}$ and k-adic relation R^k over property-domain $\boldsymbol{\alpha}$ is mapped by f into its f-image $f\alpha^k$ and fR^k, respectively, in the range of ϕ; while conversely, to every num k-tuple n^k and k-adic num relation Q^k over the range of ϕ (i.e., over the range of f) there corresponds a k-tuple $f^I n^k$ of $\boldsymbol{\alpha}$-values and k-adic $\boldsymbol{\alpha}$-relation $f^I Q^k$ which is the f-contraimage of n^k and Q^k, respectively. The notions of ‘representing’ a relation over the values of a natural variable and ‘interpreting’ a relation over the values of a formal variable may thus be explicated as

Def. 8. A relation R^k over values of a natural variable $\boldsymbol{\alpha}$ is *represented* on a num-valued formal scale ϕ for $\boldsymbol{\alpha}$ by num relation Q^k over the range of ϕ iff Q^k is the image of R^k under the scaling transformation of $\boldsymbol{\alpha}$ into ϕ.

Def. 9. A relation R^k over values of natural variable $\boldsymbol{\alpha}$ is an *interpretation* of num relation Q^k over the range of formal scale ϕ for $\boldsymbol{\alpha}$ iff R^k is the contraimage of Q^k under the scaling transformation of $\boldsymbol{\alpha}$ into ϕ.

More generally, given any mapping f of one domain into another, we can think of relations over f's argument domain as ‘represented’ by their f-images, and relations over f's range as ‘interpreted’ by their f-contraimages. In our present case, since any scaling transformation f is necessarily one-one, we know from Theorem 2 that the f-contraimage of the f-image of any relation R^k over $\boldsymbol{\alpha}$ is simply R^k itself. Hence, THEOREM 3. Num relation Q^k represents $\boldsymbol{\alpha}$-value relation R^k on scale ϕ of $\boldsymbol{\alpha}$ iff R^k is an interpretation of Q^k on ϕ.

In Part II, however, we shall encounter a situation which is formally identical with scaling in all respects except that the mapping function is not one-one, with the crucial consequence that representation and interpretation do not generally coincide.

The significance of 'interpretation' for scaling practice is, of course, that if in our study of a natural variable $\boldsymbol{\alpha}$ we have identified a property R^k over k-tuples of $\boldsymbol{\alpha}$-values which happens to interest us, and R^k is an interpretation of num-relation Q^k over the values of some formal scale ϕ for $\boldsymbol{\alpha}$, then whenever we observe that a given k-tuple n^k of nums does or does not have property Q^k we know immediately that the k-tuple of $\boldsymbol{\alpha}$-values represented on scale ϕ by n^k correspondingly does or does not have property R^k. That is,

THEOREM 4. If ϕ is a formal scale for natural variable $\boldsymbol{\alpha}$ on which num relation Q^k is interpreted by $\boldsymbol{\alpha}$-value relation R^k, then for any k-tuple α^k of $\boldsymbol{\alpha}$-values whose representation on ϕ is num k-tuple n^k, $Q^k(n^k)$ iff $R^k(\alpha^k)$.

This is actually but a corollary of the following more general representation theorem:

THEOREM 5. If $\boldsymbol{\alpha}$-value relations R^k and S^k are respectively interpretations of num relations P^k and Q^k over the values of a formal scale for $\boldsymbol{\alpha}$, then P^k entails (i.e., is a subset of) Q^k iff R^k entails S^k.

Theorems 4 and 5 show that if we have been able to find a scale ϕ for $\boldsymbol{\alpha}$ whose representation of $\boldsymbol{\alpha}$-value relation R^k is sufficiently docile formally—if, e.g., R^k's representation is a well-behaved mathematical property about which many cheerful facts are known—then study of this representation may reveal things about R^k which are much less discernible in terms of the unscaled data, such as whether or not a particular k-tuple of $\boldsymbol{\alpha}$-attributes with known scale values satisfies R^k.

There are many directions in which the concepts of scale representation and interpretation can be generalized. Most of these, which are immediate consequences of the general theory of isomorphisms, go far beyond any need which scientific research or applied technology will have for years to come. But three have sufficient immediate importance to warrant their inclusion in basic scaling theory, namely, boundary restrictions, simultaneous interpretation of multiple num relations, and conjoint interpretation of formal scales for several natural variables. The point about boundary restrictions is that some interesting relations R^k over $\boldsymbol{\alpha}$ may not be well-defined for all k-tuples in $\boldsymbol{\alpha}^k$, or R^k may coincide with the contraimage of num relation Q^k under an otherwise desirable scaling of $\boldsymbol{\alpha}$ only within a restricted subset of $\boldsymbol{\alpha}^k$. For example, if $\boldsymbol{\alpha}$ is a natural hair-color variable over mammals whose values are defined by the predicate schema 'having ____-colored hair' plus a wastebasket alternative of hairlessness, relations among hair-colors (darker-than, equally-saturated-as, etc.) which might otherwise be represented by simple arithmetic relations over a suitable numerical scale for hair-color would undoubtedly fail to extend to hair-color k-tuples including the hairlessness value. Secondly, most formal scales in practice have recognized interpretations for more than just a single relation over the scale's values. For example, 'ratio' scales are traditionally understood to be number-valued scales on which numerical ratios have an interpretation. But 'ratio' denotes not just one binary numerical relation but a transfinite class of these, one for each different value of parameter c in '____ divided by ____ equals c'. These additions—boundary restrictions and multiple interpretations—are united with a concept of 'scale interpretation' more abstractly substantival than that of Def. 9 in the following definition styled after Suppes and Zinnes (1963):

DEF. 10. An interpretation of formal scale ϕ for natural variable $\boldsymbol{\alpha}$ is a triplet $\langle \mathbf{B}, \mathbf{R}, \mathbf{Q} \rangle$ of indexically ordered sets of relations (where for each i in index set $\mathbf{i}$, 'B_i^k', 'R_i^k', and 'Q_i^k' designate a k_i-adic relation in $\mathbf{B}$, $\mathbf{R}$, and $\mathbf{Q}$, respectively) such that $\mathbf{B}$ and $\mathbf{R}$ comprise relations over $\boldsymbol{\alpha}$, $\mathbf{Q}$ comprises relations over values of ϕ, and for each Q_i^k in $\mathbf{Q}$, the corresponding B_i^k in $\mathbf{B}$ and R_i^k in $\mathbf{R}$ are such that within B_i^k, R_i^k coincides with the contraimage in $\boldsymbol{\alpha}$ of Q_i^k under the scaling transformation of $\boldsymbol{\alpha}$ into ϕ—i.e., if $\phi = f\dot{\boldsymbol{\alpha}}$,

$$(\forall \alpha^k)[B_i^k(\alpha^k) \supset [R_i^k(\alpha^k) \equiv Q_i^k(f^* \alpha^k)]]$$

DEF. 11. If $\langle \mathbf{B}, \mathbf{R}, \mathbf{Q} \rangle$ is an interpretation of scale ϕ for natural variable $\boldsymbol{\alpha}$, then the sets $\mathbf{B}$ and $\mathbf{R}$ are the interpretation's *boundary restrictions* and *content*, respectively, while the set $\mathbf{Q}$ of relations over ϕ-scale values is the interpretation's *type*. We may also say that $\mathbf{Q}$ and $\mathbf{R}$ are respectively the 'type' and 'content' of scale ϕ, relative to each other, under boundary restrictions $\mathbf{B}$.

Finally, although traditional scaling theory has thought to interpret only relations over a single scale, recent developments require a concept of 'conjoint' scaling in which num representations are found for relations over values of two or more natural variables. The easiest way to formalize this notion is by means of 'vectorial' variables:

DEF. 12. A natural variable $\mathbf{A}$ (or formal variable $\boldsymbol{\Phi}$) over domain d is an m-dimensional *vectorial* variable iff there exist m natural variables $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_m$ (formal variables $\phi_1, \ldots, \phi_m$) over $\mathbf{d}$ such that the value of $\mathbf{A}(\boldsymbol{\Phi})$ for each argument d is the property m-tuple $\langle \dot{\boldsymbol{\alpha}}_1 d, \ldots, \dot{\boldsymbol{\alpha}}_m d \rangle$ (num m-tuple $\langle \phi_1 d, \ldots, \phi_m d \rangle$).

A vectorial variable whose value for each argument d is the m-tuple $\langle \phi_1 d, \ldots, \phi_m d\rangle$ may be designated '$\langle \phi_1, \ldots, \phi_m\rangle$', while variable $\phi_i (i = 1, \ldots, m)$ is said to be the ith *component* or ith *dimension* of vectorial variable $\langle \phi_1, \ldots, \phi_m\rangle$. It is *not* assumed that all components of a vectorial variable have the same range of values.

DEF. 13. Vectorial variable $\langle \phi_1, \ldots, \phi_m\rangle$ is a formal scale for natural variable $\langle \boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_m\rangle$ iff for each $i = 1, \ldots, m$, ϕ_i is a formal scale for $\boldsymbol{\alpha}_i$. A formal scale of a natural vectorial variable may also be said to be a 'conjoint scaling' of the latter's components.

DEF. 14.[6] If $\langle \mathbf{B}, \mathbf{R}, \mathbf{Q}\rangle$ is an interpretation of formal scale $\langle \phi_1, \ldots, \phi_m\rangle$ for natural vectorial variable $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_m$, then $\langle \mathbf{B}, \mathbf{R}, \mathbf{Q}\rangle$ is a *conjoint interpretation* of scales $\langle \phi_1, \ldots, \phi_m\rangle$, the type, content, and boundary restrictions of which are $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{B}$, respectively.

The foregoing explication of 'scale interpretation' can be stretched to cover the statistical analysis of scaled data as follows:[7] A 'statistical property' of natural variable $\boldsymbol{\alpha}$ or formal variable ϕ may be defined as any abstraction from a 'distribution' of $\boldsymbol{\alpha}$ or ϕ, the latter being something which, for variables with a finite number of values, is identified by a function δ which maps the variable's values into nonnegative real numbers whose sum over δ's range is unity. Though not essential here, it should be added that the distribution of $\boldsymbol{\alpha}$ or ϕ is relative to some further condition C (notably, C specifies a particular 'population' within which the variable is distributed, together with the kind of measure—frequency vs. probability—δ is), and that if δ is the distribution function for a natural variable $\boldsymbol{\alpha}$ under condition C, then the distribution function under C for any formal scale $\phi = f\dot{\boldsymbol{\alpha}}$ for $\boldsymbol{\alpha}$ is δf^{-1}. Now, any distribution function δ whose argument domain is the range $\mathbf{v}$ of some natural or formal variable is either identical with or can be approximated as closely as we please by a distribution function δ' over $\mathbf{v}$ whose values are rational numbers with a common denominator k (where the larger is k, the closer is the approximation), while any such δ' can be represented by a k-tuple v^k_δ, of its arguments such that for each argument v_i of δ', the number of components in v^k_δ which are v_i equals k times $d'v_i$. Every rational-valued distribution function for a finite-valued variable is thus equivalent to some k-tuple of the variable's values; so if every distribution function possible for a finite valued variable is replaced with a suitably close rational approximation thereto with common denominator k,

[6]An improved version of Def. 14 would exclude from what is said to be 'jointly interpreted' any component of the vectorial scale which contributes to the interpretation only vacuously. (Component scale ϕ_i is vacuous in interpretation $\langle \mathbf{B}, \mathbf{R}, \mathbf{Q}\rangle$ of $\langle \phi_1, \ldots, \phi_m\rangle$ iff $\langle \mathbf{B}, \mathbf{R}, \mathbf{Q}\rangle$ is also an interpretation of $\langle \phi_1, \ldots, \phi_{i-1}, f\phi_i, \phi_{i+1}, \ldots, \phi_m\rangle$ for any rescaling transformation f whose range is the same as ϕ_i.)

[7]A similar though somewhat more restricted approach has previously been exploited by Adams, Fagot, and Robinson (1965)

every statistical property S abstractable from the variable's distribution becomes equivalent to some k-adic relation over the variable's values. Hence as a rational approximation, if ϕ is a formal variable derived from natural variable $\boldsymbol{\alpha}$ by scaling transformation *f*, a statistical property S of $\boldsymbol{\alpha}$'s distribution under condition C is an interpretation of statistical property T over the condition-C distribution of scale ϕ iff the *f*-contraimage of the relation over ϕ-values corresponding to T coincides with the $\boldsymbol{\alpha}$-value relation corresponding to S. To extend this treatment to variables with a transfinite number of values, we need only note that from a variable $\boldsymbol{\alpha}$ (or ϕ) of the latter sort a finite-valued variable $\boldsymbol{\alpha}'$ (or ϕ') can be derived by grouping values of $\boldsymbol{\alpha}(\phi)$ in such a fashion that for virtually all statistics which interest us, the statistical properties of $\boldsymbol{\alpha}'(\phi')$ approximate as closely as we please the corresponding statistical properties of $\boldsymbol{\alpha}(\phi)$.

Admissible Transformations and the Theory of Scale Types

I now propose to discredit certain prevalent if obscure dogmas, emanating largely from the views of S. S. Stevens, concerning what can and what cannot legitimately be done with a num-scaled variable. The proportions in which my summary of these strictures reflects, respectively, Stevens' own beliefs, exegeses by his followers, and my own distorted perception of these, are of no great moment; for the destruction even of straw men, if sufficiently fearsome, can present a sobering warning to those who have become enamored of strawish ways.

Basic to the dogmas now at issue are the notions of *scale type* and *admissible transformation*. A scale's 'type' in the Stevens' tradition is basically the same as defined here, namely, a relation or set of relations over the scale's values which represents an empirical relation or set of empirical relations over the data scaled; while an 'admissible' transformation is a rescaling transformation under which this interpretation is preserved—i.e., if ϕ is a numerical scale of type $\mathbf{Q}$ with respect to content $\mathbf{R}$, then transforming ϕ into another scale ϕ' is admissible if and only if number relations $\mathbf{Q}$ still represent empirical relations $\mathbf{R}$ on scale ϕ'. It is then alleged that a rescaling ϕ' of ϕ is a methodologically legitimate alternative to scale ϕ if and only (with emphasis upon the 'only') if ϕ' is related to ϕ by an admissible transformation, and that a statistic or other relational property over scale values is 'meaningful' if and only if that property is invariant under all admissible transformations of the scale. We shall now see that both of these claims are complete nonsense.

To begin, it is important to be clear about a serious defect in the Stevens-tradition (hereafter abbreviated 'ST') conception of 'scale type'. Whereas we have here defined a scale's type as relative to a particular interpretation, ST doctrine construes this as an absolute property of the scale—i.e., scale ϕ either is or is not of type $\mathbf{Q}$, simpliciter. In particular, ST proponents are especially industrious at proscribing scales as *not* being of such-and-such a type, notably when scale-type labels are applied in the exclusive sense by which, e.g., a scale identified as 'ordinal' is thereby also stigmatized as *not* having an 'interval' or 'ratio' interpretation. In general, whenever a set $\mathbf{Q}$ of num relations is said to be *the* type of num-valued scale ϕ, it is implied that no interpretation exists on ϕ for num relations which are not entailed by the set $\mathbf{Q}$. The fallacy in this, however, is simply that

THEOREM 6. If ϕ is any arbitrary formal scale for natural variable $\boldsymbol{\alpha}$, every relation over values of $\boldsymbol{\alpha}$ is represented by some relation over values of ϕ, while conversely, every relation over ϕ-values is interpreted by some $\boldsymbol{\alpha}$-value relation. *Corollary*: For any arbitrary set $\mathbf{Q}$ of relations over values of an arbitrary formal scale ϕ, there exists a scale interpretation with respect to which ϕ is of type $\mathbf{Q}$.

For if f is any scaling transformation with range $\mathbf{n}$ such that $\phi = f\dot{\boldsymbol{\alpha}}$ while R^k and Q^k are arbitrary relations over $\boldsymbol{\alpha}$ and $\mathbf{n}$, respectively, then R^k is represented on scale ϕ by its f-image in $\mathbf{n}$ while the interpretation of Q^k in $\boldsymbol{\alpha}$ is Q^k's contraimage. In particular, Q^k holds for num k-tuple $\langle n_1, \ldots, n_k \rangle$ iff the $\boldsymbol{\alpha}$-values represented by $n_1, \ldots, n_k$ respectively, satisfy the relation 'are respectively represented on scale ϕ by nums standing in relation Q^k'. (Similarly, any statistical property S of ϕ's distribution under condition C has at least the interpretation described by 'The distribution of $\boldsymbol{\alpha}$ under condition C is such as to give its ϕ-scale representation property S'.) Thus, Brainsweat U.'s numerical scale for the natural College-major variable is an ordinal scale when numerical inequalities are interpreted as 'majoring in ____ has a larger College-major-score scale value than does majoring in ____', an interval scale when numerical difference comparisons are interpreted as 'majoring in ____ has a C.m.s. scale value which is closer to that of majoring in ____ than to that of majoring in ____', and a ratio scale when numerical ratios are interpreted as 'majoring in ____ has a C.m.s. scale value c times as large as the C.m.s. scale value of majoring in ____'.

Admittedly, that Psychology majoring has a larger C.m.s. scale value than Chemistry majoring does not seem to be a very *significant* fact about these two college majors, since this inequality derives from nothing more than an arbitrary choice of labels by the Brainsweat U. administration. But crucial as this intuitive sense of 'significance' is for the methodology of scale interpretation, its analysis lies forever beyond the grasp of set-theoretical approaches to scaling; for in extensional logic no one relation on a given domain differs in ontological kind from any other. For example, there is no inherent extensional inequity of significance between 'is female' on the one hand and 'was born on an even-numbered calendar date' on the other, since both are subsets, roughly equal in size, of the domain of living organisms, and differ only in not being coextensive. Similarly, if John Smith holds ACME Life Ins. policy No. 12345678 and is also the only person ever to have regrown his head after it was severed in an accident, holding-ACME-policy-No.-12 345 678 and having-regrown-an-accidently-severed-head must have exactly the same set-theoretical significance, for extensionally they are the very same property, namely the unit-class of John Smith. Considerations such as these lead implacably to the conclusion that any serious attempt to determine which num relations over the values of formal scale have 'significant' interpretations must explore in technical depth many profound problems in epistemology, semantics and the philosophy of science about which scaling theory to date has not dared even to whisper. As for myself, I am sceptical that there exists any useful dichotomous distinction between properties which are 'significant' and those which are not[8]; instead, I suspect that each property or relation which we are able to conceive has its own particular degree of significance or 'meaningfulness' imparted by its nomological implications and/or the personal satisfactions we receive from it. If so, then the problem of interpretation for scaling theory is not to say *whether* a given scale's content is significant but *what* its significance is. To keep Theorem 6 from scuttling the ST theses about scale types out of hand, however, let us suppose that we have somehow managed to define a sense in which some but not all the relations over values of a natural variable are 'significant'. Then we could introduce a non-vacuous absolute concept of scale 'type' as follows:

DEF. 15. A scale ϕ for natural variable $\boldsymbol{\alpha}$ is (absolutely) of S-type $\mathbf{Q}$ iff $\mathbf{Q}$ is a set of relations over ϕ-values such that the contraimage in a of every relation in $\mathbf{Q}$ under the scaling transformation f of $\boldsymbol{\alpha}$ into ϕ is 'significant'. Scale ϕ is *exhaustively* of S-type $\mathbf{Q}$ iff ϕ is of S-type $\mathbf{Q}$ and there is no 'significant' relation over $\boldsymbol{\alpha}$-values whose ϕ-image is not included in (or at least entailed by) the set of num relations $\mathbf{Q}$.

(The prefix in 'S-type' may be read either as abbreviation for 'Stevens' or as a parameter which takes different values for different senses of 'significant'. In what follows, I shall not consistently make this prefix explicit.)

Actually, the ST concept of scale 'type' has within recent years become even more tenuous than is suggested by the difficulty in giving Def. 15 cash value, for ST adherents increasingly characterize scale type by admissible transformations rather than interpreted num relations. In terms of Def. 15, the notion of an admissible transformation may be made precise as follows:

DEF. 16. If ϕ is a scale of S-type $\mathbf{Q}$, then a rescaling transformation g of ϕ into

[8] At one time I thought that a concept of 'naturally significant' relations over values of a natural variable could be developed by excluding those which derive from a semantic scaling of the variable. Further exploration of this possibility, however, has pretty well convinced me that the task is hopeless.

scale $\phi' = g\phi$ is *admissible* iff, for every relation Q^k in $\mathbf{Q}$, $g^*Q^k = Q^k$. [9]

Then to each S-type $\mathbf{Q}$ there corresponds a set $\mathbf{g_Q}$ of functions such that a rescaling transformation g belongs to $\mathbf{g_Q}$ iff g is an admissible transformation of scales of S-type $\mathbf{Q}$, and it is easy to slip over into thinking of $\mathbf{g_Q}$, rather than $\mathbf{Q}$, as definitive of scale 'type'. But it is precisely in its views on admissible transformations that ST doctrine becomes most questionable. Observe to begin with that a class of admissible transformations is defined *in terms of* a scale ϕ and its class $\mathbf{Q}$ of significantly interpretable num relations. Hence the admissible-transformation concept says nothing about what *initial* scale ϕ to adopt for a natural variable $\boldsymbol{\alpha}$—it only says that once we have scaled $\boldsymbol{\alpha}$ as ϕ, *then* we are allowed to rescale only by transformations which preserve its type. But the set of significant $\boldsymbol{\alpha}$-value relations is represented by some set of num relations on *any* num-valued formal scale for $\boldsymbol{\alpha}$ (cf. Theorem 6), so if ϕ' and ϕ are two alternative scales for $\boldsymbol{\alpha}$ related by an inadmissible transformation, a person who first chose to scale $\boldsymbol{\alpha}$ as ϕ' would be prohibited by ST doctrine from subsequently replacing ϕ' by scale ϕ even while ϕ' is proscribed for use by anyone who first scaled $\boldsymbol{\alpha}$ as ϕ.

If $\boldsymbol{\alpha}$-value relations $\mathbf{R}$ are represented on scale ϕ by num relations $\mathbf{Q}$, and ϕ is rescaled as ϕ' by transformation $g' = g\phi$, then g maps $\mathbf{Q}$ into set $g^*\mathbf{Q}$, each relation $g^*Q_i^k$ in which represents on scale ϕ' the very same $\boldsymbol{\alpha}$-value relation as does relation Q_i^k on scale ϕ. If g is not an admissible transformation, then in general $g^*Q_i^k \neq Q_i^k$, but this does not imply any sacrifice in going from ϕ to ϕ', not even necessarily of pragmatic convenience; it only means that rote habits for interpreting scores on ϕ cannot be transferred without modification to scores on ϕ'. We neither gain nor lose 'meaningfulness' by inadmissible scale transformations; we merely make an adjustment in *which* features of our scaled data carry its natural significance. Any two scales for the same natural variable have the same *content* even when their *types* differ.

I strongly suspect that admissible-transformation strictures are grounded upon

[9]The concept of admissible transformation is seldom put quite this strongly. In most of the ST literature, the number relations composing a scale's 'type' are conceived as subsets of k-tuples of *all* numbers, not just of the scale's range as stipulated by Def. 15; whence if 'admissible transformations' are defined to be those which leave scale type invariant, ϕ' can be derived from ϕ by an admissible transformation even when ϕ and ϕ' have different ranges, namely, when the f-image and f'-image of each content relation R^k (where f and f' are the transformations which scale the natural variable in question as ϕ and ϕ', respectively) are both subsets of the same more inclusive relation Q^k over the full number domain. But the 'type' of scale ϕ with respect to content R^k is then hopelessly nonunique, with the class of 'admissible' transformations correspondingly ill-defined; for R^k's f-image f^*R^k is a subset of every relation $f^*R^k \cup T^k$, and ϕ correspondingly of 'type' $f^*R^k \cup T^k$, such that T^k is any arbitrary set of k-tuples comprising only numbers not included in ϕ's range. Even so, none of the criticisms brought here against admissible-transformations doctrine depend upon scale 'type' being defined so restrictively as in Def. 15.

an unconscious belief that a given natural relation can be represented by at most one numerical scale relation. As we have seen, this is wildly untrue; but it does, nonetheless, point to a legitimate scaling concern known as the 'representation problem'. While every $\boldsymbol{\alpha}$-value relation has some num representation on every given formal scale for $\boldsymbol{\alpha}$, it is *not*, conversely, the case that for any $\boldsymbol{\alpha}$-value relation R^k and num relation Q^k there exists a formal scale for $\boldsymbol{\alpha}$ on which Q^k represents R^k. Further, given a set $\mathbf{R}$ of $\boldsymbol{\alpha}$-value relations and a correspondingly ordered set $\mathbf{Q}$ of num relations, there may exist for each R_i^k in $\mathbf{R}$ separately a class of scales for $\boldsymbol{\alpha}$ on which R_i^k is represented by the corresponding Q_i^k in $\mathbf{Q}$ even while there is no single scale for $\boldsymbol{\alpha}$ on which all relations in $\mathbf{R}$ are simultaneously represented by the corresponding relations in $\mathbf{Q}$. (To cite an example which frequently arises in practice, we may have a choice between a scale which represents equal $\boldsymbol{\alpha}$-differences by equal number differences and one which represents $\boldsymbol{\alpha}$'s distribution by a normal curve, but have no scale which does both.) The 'representation problem', then, is to devise effective means for judging whether the $\boldsymbol{\alpha}$-value relations in a given set $\mathbf{R}$ can be simultaneously represented by the num relations in a given set $\mathbf{Q}$, and if so, how to compute a scale on which this representation holds. Although advanced scaling theory has devoted considerable study of this sort to certain specific sets of num relations, admissible-transformations doctrine has been an impediment to this movement's most powerful development. For it is possible to simultaneously represent the natural relations in a set $\mathbf{R}$ by the num relations in a set $\mathbf{Q}$ if and only if it is also possible to represent $\mathbf{R}$ by the transformed set $g^*\mathbf{Q}$, where g is *any* rescaling transformation applicable to the scale on which $\mathbf{Q}$ represents $\mathbf{R}$. Let the *representation group* of a scale type $\mathbf{Q}$ be defined as the set of all scale types into which $\mathbf{Q}$ can be mapped by rescaling transformations. Then to solve the representation problem for one scale type is also to solve it for all others in the same representation group, and the most penetrating question to ask when considering prospective scales on which to represent $\boldsymbol{\alpha}$-value relations $\mathbf{R}$ is not whether content $\mathbf{R}$ can be represented by scale type $\mathbf{Q}$, but to what representation group with respect to content $\mathbf{R}$ must a scale—any scale—for $\boldsymbol{\alpha}$ belong.

I urge, therefore, that both scaling theory and scaling practice stand to profit immensely more from study of representation groups than from continued concern for type-preserving transformation groups. For theory, determining the abstract structure common to all scale types in a given interpretation group concomitantly reveals what structure a set of natural relations must have to be representable on any scale by a set of num relations in this group. And for practice, it is desirable not merely that we remain uninhibited by admissible-transformation taboos, but also that we become aware of specific alternatives for practical representation. For while the overwhelming majority of scale types in any given representation group are hopelessly chaotic analytically, the group is bound to contain one or more types just as manageable, or nearly so, as the one which is our first choice. For example, on the Mohs scale of mineral hardness, content $\mathbf{R} = \langle$is-softer-than, is-just-as-hard-as, is-harder-than$\rangle$ is represented by the ordered set of number relations $\mathbf{Q} = \langle$is-less-than, equals, is-greater-than$\rangle$. But this content could just as well be represented by set $\mathbf{Q}' = \langle$is-greater-than, equals, is-less-than$\rangle$, which is in the same representation group as $\mathbf{Q}$ even though the transformation of the Mohs scale needed to represent $\mathbf{R}$ by $\mathbf{Q}'$, namely an inversion of the scale order, is not 'admissible' insomuch as $\mathbf{Q}' \neq \mathbf{Q}$. (The fact that the relations in $\mathbf{Q}'$ are a permutation of the relations in $\mathbf{Q}$ does not change the fact that neither is-less-than nor is-greater-than is invariant under order inversion.) Being alert to workable scale-type alternatives is especially important when we have several natural relations for which simultaneous docile representation is desired. For example, if the values of natural variable $\boldsymbol{\alpha}$ sustain a quadratic relation R^4 describable as 'the superiority of α_h over α_i is less than the superiority of α_j over α_k' and representable on scale ϕ for $\boldsymbol{\alpha}$ by numerical relation $n_h/n_i < n_j/n_k$, while the variable's distribution is skewed on scale ϕ but can be normalized by a logarithmic transformation, then we will have both a normal scale distribution and a tractable representation of R^4 by numerical relation $n_h - n_i < n_j - n_k$ if ratio scale ϕ is 'inadmissibly' converted into an interval scale by the log transform. (That ratio scales and positive-valued interval scales are of types belonging to the same representation group has become reasonably well recognized today, but ST adherents have apparently not as yet appreciated the devastation this wreaks upon the concept of 'admissible transformation'.)

Its preoccupation with the mathematics of invariance has also led admissible transformations doctrine to prescribe which statistical properties of an admissibly scaled distribution are legitimate and which are not. Specifically, if ϕ is a numerical scale of type $\mathbf{Q}$, a statistic S abstracted from ϕ's distribution is considered meaningful if and only if it is invariant under all admissible transformations of ϕ. Since statistical properties are essentially a class of relations over the scale's values (see above), this coupling of statistical 'meaningfulness' with invariance under admissible transformations is tantamount to the thesis that a relation P^k over values of a scale of type $\mathbf{Q}$ represents a significant content relation if and only if every rescaling transformation which maps each relation in $\mathbf{Q}$ into itself does the same for P^k. But the only support which, to my knowledge, this thesis has ever received is the suggestion by Adams, Fagot and Robinson (1965) that if ϕ is a scale for natural variable $\boldsymbol{\alpha}$ on which the num relations in set $\mathbf{Q}$ respectively represent the $\boldsymbol{\alpha}$-value relations in set $\mathbf{R}$, then a num relation P^k is mapped into itself by all admissible transformations of scale type $\mathbf{Q}$ iff P^k represents on ϕ an $\boldsymbol{\alpha}$-value relation which is definable in terms of the relations in $\mathbf{R}$. Now it can indeed be proved that if there exists a rescaling transformation g such that $g^*\mathbf{Q} = \mathbf{Q}$ while $g^*P^k \neq P^k$, then a predicate designating the $\boldsymbol{\alpha}$-value relation represented on ϕ by P^k cannot be constructed within the first-level predicate calculus from predicates designating the relations in **R**. But this is entirely compatible with the possibility that the $\boldsymbol{\alpha}$-value relation represented on ϕ by P^k is 'significant' even though it cannot be defined in terms of relations R. The proper direction of argument here is not "Statistic S is not meaningful for a scale of type **Q** because it is not invariant under all admissible transformations for type-**Q** scales", but rather, "This scale cannot be exhaustively of type **Q** because statistic S has a meaningful interpretation on it even though S is not invariant under admissible transformations of type-**Q** scales". Moreover, even if ϕ-scale relation P^k is mapped into itself by all admissible transformations for scales of type **Q**, it does not follow that the content of ϕ represented by P^k can be meaningfully defined in terms of the contents represented by **Q** in any intuitively acceptable sense of 'meaningful'; for otherwise it would follow that if scale ϕ for natural variable $\boldsymbol{\alpha}$ is of a type **Q** which has no admissible transformations other than the Identity transform (which can easily occur even when **Q** comprises merely a singly binary relation)[10], then every arbitrary set of k-tuples of $\boldsymbol{\alpha}$-values is 'meaningfully' definable in terms of the $\boldsymbol{\alpha}$-value relations represented on ϕ by **Q**.

ST scale-type and admissible-transformations dogma is not merely otiose, it is pernicious as well. It is pernicious because it systematically confuses number relations with natural ones and suppresses any attempt to learn what significant relations in fact hold for the values of particular natural variables under study. Whatever Stevens himself may think of this, the pattern of scale-interpretive reasoning that ST adherents habitually employ is to argue that number relation P^k is or is not 'meaningful' on scale ϕ *because* scale ϕ is of type **Q**, while it is P^k itself which is correspondingly thought to be meaningful rather than its natural contraimage under the scaling transformation. For example, the claim is often made that an IQ of 150 cannot be said to be twice as great as an IQ of 75 because the IQ scale has no fixed zero (i.e., because shifts in the scale's origin are admissible and hence that the scale is not of the ratio type). But the reason *why* the IQ scale has no fixed—or rather, meaningfully fixable—zero point is that intelligence has no known feature which can be represented by a numerical scale property whose transformational invariance requires a fixed zero.[11] That is, if we knew how to interpret 'IQ score x is twice as large as IQ score y', this would give

[10] Though to mention it is rather like kicking a cripple, it should not go unnoted that ST doctrine implies that any scale type **Q** which includes a 'nominal' interpretation of scale values has no admissible transformations other than Identity. For if R_i^1 is the monadic num relation 'is equal to n_i' and this is interpreted on scale ϕ of natural variable $\boldsymbol{\alpha}$ as 'is the property α_i, then any rescaling transformation which maps R_i^1 into itself must leave the scale representation of α_i unchanged. (Obviously ST adherents have always *intended* to allow as 'permissible' all one-one rescalings of nominal scales, but how this can be made consistent with the body of admissible-transformation beliefs still remains to be shown.)

[11] That is, excluding the 'nominal' interpretation of scale values, which prohibits any rescalings at all under the 'admissible transformations' egis (cf. fn. 10).

the IQ scale a fixed zero point. On the other hand, if we were to believe that IQ is a ratio scale and *inferred* from this that 'IQ score x is twice as large as IQ score y' is meaningful, then, insomuch as the latter is but a relation between numbers which represent different degrees of intelligence it would still remain to identify the binary intelligence relation which holds for any two intelligence levels when the corresponding IQ numbers stand in ratio $2:1$. In this way, the ST perspective disastrously inverts the proper order of inquiry by seeking to answer questions about scale-property meaningfulness in terms of the scale's type rather than by judging a scale's type in terms of what on it is meaningful, and stultifies concern for scale content by confusing the question of *whether* relation P^k over values of scale ϕ is meaningful with the question of *what* it means. To add final insult, when scale type is identified in terms of an admissible-transformation group $\mathbf{g}$, rather than by a particular set of scale relations whose admissible transformations constitute $\mathbf{g}$, merely stating scale type does not specify even what numerical scale relations have significant interpretations (since two different sets of scale relations can define the same admissible transformations), much less what these interpretations may be.

In summary, then, I submit—loudly—that there is no such thing as an illicit scaling procedure. One scale for a given scientific variable can be at worst merely less convenient than another. If we know of some content relation which *can* be represented by an especially well-behaved numerical relation, then, other desiderata equal, it would be silly not to capitalize on this by selecting our scale accordingly; but this is strictly a matter of personal taste, not of what *ought* to be, and if we choose to sacrifice one scaling convenience to achieve another, no one choice is any more 'correct' than any effectively definable alternative to it. Neither are there any illegitimate statistics or other numerical properties of scaled data, no matter how arbitrarily a particular scale may have been defined. At all times, the proper question to ask about a given scale property is not whether it means anything, but *what*. If we can but find some interpretive significance in a statistic proscribed for scales of the type to which the scale in question has been deemed to belong, then that scale's 'type' therewith broadens[12] to accommodate this newfound content; whereas so long as we know not what a given statistic signifies about the unscaled data, no anticipatory glee over how that statistic is guaranteed significance by the scale's type will do a thing to ameliorate its present *de facto* meaninglessness for us.

In short, the alpha and omega of scaling is *significant content*. If we have

[12]Should we say that scale types in a series such as 'nominal', 'ordinal', 'interval' are increasingly *broad* or increasingly *narrow*? The question is not as trivial as it first seems, for in view of the inverse relation between the number of relations in a scale type $\mathbf{Q}$ and the number of $\mathbf{Q}$'s admissible transformations, a person's answer to it reveals whether he thinks of interpretable scale properties in terms of admissible transformations, or admissible transformations in terms of interpretable scale properties.

it, then we can formally process it in whatever fashion suits our whim. And if we do not, no incantations about scale types and admissible transformations will provide it for us. But what sorts of contents are there for a scale to have? Is it possible to develop any systematic ideas about general *kinds* of significant relations which the values of a natural variable may make available for scale representation? Does there conceivably exist a taxonomy for natural variables in which a variable's content category reveals something about the character and prevalence of laws in which this variable is likely to participate? Can we devise any useful systems of definitional schemata which algorithmically generate significant scale content for all the num relations in a certain set $\mathbf{Q}$ as soon as significant content has been found for a suitable subset of $\mathbf{Q}$? Scaling theory has so far scarcely even thought about such questions, let alone provided any hints of an answer. The final sections of this paper attempt to move in this direction, but they are little more than a hasty scouting of the terrain.

2. Pattern Analysis and Factorial Decomposition

One reason why the theory of scale content has remained so underdeveloped is measurement theory's failure to examine any substantive examples of this. For scarcely any of the data-system representations studied so profusely in recent years are instances of *scaling* at all; instead, these illustrate a very different methodological procedure which I shall call 'factorial decomposition'. That the distinction between scaling and factorial decomposition has not heretofore been properly appreciated is perhaps understandable, for their formal theories differ only in one or two subtle technicalities and in some applications essentially coalesce. Yet whereas scaling methodology rests upon nothing more profound than human convenience, factorial decomposition lies at the most advanced frontiers of scientific inference.

The Concept of 'Data Pattern'

The locus of factorial decomposition in substantive research is at the point of convergence between *data models* and *data patterns*. A 'model' in this context is a hypothesis about nomic regularities and source variables tentatively entertained as a more-or-less idealized explanation for some configuration of observed data. For example, if the matrix of empirical correlations among various tests of intellectual functioning can be fairly well approximated in all its off-diagonal entries by a matrix of unit rank, we might adopt as a model of these intercorrelations the Spearman 'two-factor' hypothesis that each test variable derives from a single source variable common to all, plus a specific factor unique to that particular test. A *pattern*, on the other hand, is some configural property which the data actually have—i.e., which can be analytically abstracted from the data as given. Thus if the off-diagonal entries in an empirical correlation matrix are well approximated by a matrix of unit rank, this is itself a higher-order datum independent of any conjectures we may or may not be willing to countenance about the source of this observed structure. In short, 'pattern' is something which the data *manifest*, whereas a 'model' proposes *why* the data have a particular pattern.

While this is no occasion for a general review of pattern analysis, I shall work my way to the concept of factorial decomposition through a series of definitions which suggest directions for the thrust of a more comprehensive account. We begin with the generic concept of 'data pattern', which may be taken to denote any property of data. To explicate this extensionally, let a set $\mathbf{p}$ of mutually consistent propositions be called a *possible data configuration* when each element p_i in $\mathbf{p}$ describes a state of affairs which has been, may be, or might have been discovered to be an empirical fact, while *data space* is the set of all possible data configurations. Two possible data configurations $\mathbf{p}$ and $\mathbf{p}'$ are *alternatives* if they are jointly inconsistent while every consequence of the propositions in $\mathbf{p}$ is either entailed by or is incompatible with the propositions in $\mathbf{p}'$ and conversely. Then in extensional terms, a *data pattern* is some subset of data space, while a *pattern variable* is any natural or formal variable whose argument domain is data space or some subset thereof such that a given possible data configuration $\mathbf{p}$ is in the pattern variable's argument domain iff this domain also includes all alternatives to $\mathbf{p}$.

The abstract notions of 'data pattern' and 'pattern variable' subsume a great many data properties of the utmost triviality, and disclosure of what differentiates scientifically instructive from inferentially empty data patterns (or, more penetratingly, classification of data patterns according to the types of inferences they support) is one of the more important tasks ahead for the theory of data analysis. On the other hand, not only do statistical properties of the more familiar sort count as 'data patterns' in the present sense, but anything which can be said about data configuration $\mathbf{p}$ *vis-à-vis* a set $\mathbf{H}$ of alternative explanatory hypotheses or models from a hypothesis-testing perspective can also be rephrased in pattern-analytic terms. Thus the 'likelihood function' for $\mathbf{p}$ in $\mathbf{H}$—i.e., the function over $\mathbf{H}$ whose value for a given argument H_i—is the probability (or probability density) of $\mathbf{p}$ given H_i—is the value for $\mathbf{p}$ of a certain 'functional' (i.e., a function-valued function) determined by model-set $\mathbf{H}$ whose argument domain is some subset of data space including $\mathbf{p}$ and all its alternatives; while the functional which maps $\mathbf{p}$ and its alternatives into a distribution of posterior probabilities over the hypotheses in $\mathbf{H}$ is likewise a 'pattern variable' in the present sense.

Conditional probabilities of data configurations given alternative models, or posterior probabilities of models given the data, are one way in which data patterns are defined by data models, but considerably more fundamental is the sort of pattern which is at issue when we speak of a model as 'fitting', or failing to fit, a given data configuration. We may say (a) that a model M *fits* a possible data configuration $\mathbf{p}$ iff $\mathbf{p}$ is logically consistent with M; (b) that the content of $\mathbf{p}$ accounted for by a model M which fits $\mathbf{p}$ is the set of all propositions entailed both by $\mathbf{p}$ and by M; and (c) that a model M 'projects' data pattern P iff, for every possible data configuration $\mathbf{p}$, M fits $\mathbf{p}$ only if $P(\mathbf{p})$. Further, to recognize the custom of building flexibility into models by means of adjustable parameters, we may call a model M *open* when there corresponds to M a set $\mathbf{m}$ of models such that (i) M fits a possible data configuration $\mathbf{p}$ iff there exists an m in $\mathbf{m}$ which fits $\mathbf{p}$, and (ii) there are at least two models in $\mathbf{m}$ such that not every possible data configuration which fits the one also fits the other. (Clause (ii) is to prevent the definition from applying trivially to every model.) If m and M are two models such that M is open, m is not open, and every possible data configuration which fits m also fits M, then m is a closure of M; while to find a *solution* of model M for data configuration $\mathbf{p}$ is to identify a closure of M which fits $\mathbf{p}$.

Excluding stochastic models, it is seldom that we are able to find *exact* solutions of scientifically interesting models for real data. In practice, therefore, it becomes important to be able to judge not merely whether or not model M fits data configuration $\mathbf{p}$, but *how well*. Two distinct styles for accomplishing this have emerged to date. The *approximation-error* approach devises a measure of how closely each closed (i.e., not open) model m *approximately* fits a data configuration $\mathbf{p}$, usually in terms of the difference between $\mathbf{p}$ and the possible data configuration most similar to $\mathbf{p}$ which is fitted exactly by m, and then considers the best approximational solution of open model M for $\mathbf{p}$ to be that closure of M which maximizes this measure. In contrast, the *stochastic-error* approach begins with core models which could be fitted to data by an approximation error method but instead augments these with open stochastic components (i.e., an addendum of hypothesized 'error' or 'unique' variables) in virtue of which the augmented model has an exact solution for every possible data configuration. How well such a model fits data configuration $\mathbf{p}$ is then assessed by the likelihood of its stochastic solution for $\mathbf{p}$. The essential difference between approximation-error and stochastic-error models thus lies in their treatment of the disparity between extant data and ideal patterns which the data manifest only roughly, the former viewing this as error *of* the model while the latter puts the 'error' *in* the model as an additional hypothesized ingredient of reality. These two approaches develop the same nonstochastic core into rather different data-processing algorithms, and which one brings the data's structure into sharper focus is still another of research methodology's many unexplored questions. Here, however, we shall ignore the further complexities of less-than-ideal data patterning by presupposing an exact fit of model to data regardless of how this is accomplished.

Factorial Decomposition Patterns

Virtually all models of the 'explanatory' sort postulate certain hypothetical variables to underlie the data variables in such fashion that values on the former can be found for the latter's arguments which, by virtue of the assumed nomic regularities, imply some or all of the data to which the model is fitted. Further, in applications of factorial decomposition models, each datum proposition asserts or denies that certain objects are related in a certain way, where these 'objects' need not all be of the same ontological kind. For example, the data configuration might consist of propositions of form 'Person j has score c on test t' and 'Person j judges stimulus s_1 to resemble stimulus s_2 more than [the same as, less than] it does stimulus s_3', in which the 'objects' are of three kinds—persons, tests, and stimuli. A factorial decomposition model of these data would then hypothesize that these persons, tests, and stimuli are all positioned in an underlying 'genotypic' or 'source-variable' space such that which score person j gets on test t, and which similarity-judgment person j makes of stimulus triad $\langle s_1, s_2, s_3 \rangle$, is determined in a specified way by the genotypic coordinates of objects j, t, s_1, s_2, and s_3.

To express this notion more precisely, suppose that each proposition in data configuration $\mathbf{p}$ ascribes some relation R_i^k in set $\mathbf{R}$ to some k-tuple o^k of objects from domain $\mathbf{o}$. Then successful fitting of a factorial decomposition model to $\mathbf{p}$ consists in finding a mapping ψ of $\mathbf{o}$ into a num domain $\mathbf{n}$ in such fashion that to each R_i^k in $\mathbf{R}$ there corresponds a num relation Q_i^k of a certain specified kind such that for every k-tuple o^k of data objects whose status on R^k is determined by $\mathbf{p}$, $Q_i^k(\psi^* o^k)$ iff $R_i^k(o^k)$. To formalize this situation with explicit reference to models and data propositions (which are semantical structures), however, is unnecessarily cumbersome for present needs. The fact that a certain factorial decomposition model can be fitted to a given data configuration $\mathbf{p}$ reveals that the relations cited in $\mathbf{p}$ conform to the data pattern projected by the model, at least over the restricted domain of objects to which they are ascribed in $\mathbf{p}$. Factorial decomposition can hence be defined at the object-language level as follows:

Def. 17. A factorial decomposition of a set $\mathbf{R}$ of empirical relations over object domain $\mathbf{o}$ is a triplet $\langle \psi, \mathbf{B}, \mathbf{Q} \rangle$ in which ψ is a function from $\mathbf{o}$ into a num domain $\mathbf{n}$, $\mathbf{B}$ is a set of relations over $\mathbf{o}$, $\mathbf{Q}$ is a set of relations over $\mathbf{n}$, and $\mathbf{R}$, $\mathbf{B}$, and $\mathbf{Q}$ are correspondingly ordered by an index set $\mathbf{i}$ in such fashion that for every $i \in \mathbf{i}$

$$(\forall o^k)[B_i^k(o^k) \supset [R_i^k(o^k) \equiv Q_i^k(\psi^* o^k)]]$$

i.e., each R_i^k in $\mathbf{R}$ is so represented by its corresponding Q_i^k in $\mathbf{Q}$ that for every k-tuple o^k of empirical objects qualifying under boundary conditions B_i^k, whether or not o^k satisfies empirical relation R_i^k is determined by whether or not the k-tuple of nums into which ψ maps o^k satisfies num relation Q_i^k.

DEF. 18. If $\langle\psi, \mathbf{B}, \mathbf{Q}\rangle$ is a factorial decomposition of empirical relations $\mathbf{R}$, then the sets $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{B}$ are the decomposition's *type*, *content*, and *boundary restrictions*, respectively, while ψ is its *factorization function.*

That is, within its boundary restrictions, the content of a factorial decomposition coincides with the contraimage of its image under the factorization function. Since the latter is not stipulated to be one-one, this within-$\mathbf{B}$ congruence of $\psi^I\psi^*\mathbf{R}$ with $\mathbf{R}$ reveals a pattern in the data which is *not* generally a mathematical consequence of Theorem 2.

A closed factorial decomposition model of type $\mathbf{Q}$, content $\mathbf{R}$, and boundary restrictions $\mathbf{B}$ is a hypothesis about the nomic sources of relations $\mathbf{R}$ which implies the existence of a factorization function ψ such that $\langle\psi, \mathbf{B}, \mathbf{Q}\rangle$ is a factorial decomposition of relations $\mathbf{R}$. An open factorial decomposition model of type Ω and boundary restrictions Γ for $\mathbf{R}$ is correspondingly a hypothesis which implies the existence of a function ψ, natural relations $\mathbf{B}$, and num relations $\mathbf{Q}$ such that $\langle\psi, \mathbf{B}, \mathbf{Q}\rangle$ is a factorial decomposition of $\mathbf{R}$ in which $\mathbf{B}$ and $\mathbf{Q}$ satisfy restrictive conditions Γ and Ω, respectively. On the other hand, 'is compatible with the existence of a factorial decomposition of type Ω for relations $\mathbf{R}$' is a predicate over possible data configurations which asserts nothing that cannot be confirmed or denied of an argument $\mathbf{p}$ by analytic abstraction from $\mathbf{p}$, and which hence describes a data *pattern* which holds (if it does) for $\mathbf{p}$ irrespective of the plausibility of any particular *model* for $\mathbf{R}$ of this type.

Def. 17 may be applied to a system $\mathbf{R}$ of relations holding only between 'objects' of different kinds by letting $\mathbf{o}$ comprise all objects, irrespective of kind, which are arguments of the to-be-decomposed relations, but allowing only k-tuples of the proper object-kind mixture to qualify under the boundary restrictions.

Examples

To clarify the foregoing abstractions and illustrate their scope, I shall briefly cite several instances of factorial decomposition which have already received considerable investigation in substantive science. It will be noted that in most of these, the 'nums' into which the empirical objects are mapped are not just single numbers but are more complex numerical constructions.

Linear factor analysis. Suppose that $\mathbf{t}$ is a battery of tests, upon each of which every person in a population $\mathbf{j}$ has a known numerical score. Then it may be possible to assign an m-component vector of real numbers to each person j in $\mathbf{j}$ and to each test t in $\mathbf{t}$ in such fashion that the inner product of the vectors assigned to person j and test t, respectively, equals j's score on t. The empirical relational system in this instance is the set $\mathbf{R} = \{$The score of person ____ on test ____ is $c\}$, in which parameter c is also the ordering index, while for each person or test k in combined object domain $\mathbf{j} \cup \mathbf{t}$ there exists a vector $\langle x_{1k}, \ldots, x_{mk} \rangle$ of real numbers such that for any person j in $\mathbf{j}$ and test t in $\mathbf{t}$, $\langle j, t \rangle$ satisfies R_c^2 (i.e., j's score on t is c) iff $\sum_{i=1}^{m} x_{ij} x_{it} = c$. Hence in this case if ψ is the function over $\mathbf{j} \cup \mathbf{t}$ which maps each $k \in \mathbf{j} \cup \mathbf{t}$ into its vector $\langle x_{1k}, \ldots, x_{mk} \rangle$ while P_c^2 is the relation comprising all pairs of m-component real-number vectors whose inner product equals c and relation B_c^2 over $j \cup t$ is satisfied by object-pair $\langle j, t \rangle$ iff $j \in \mathbf{j}$ and $t \in \mathbf{t}$, then $\langle \psi, \{B_c^2\}, \{P_c^2\} \rangle$ is a factorial decomposition of { ____'s score on ____ is c}.

A simpler though more superficial way to describe the data-analytic character of linear factor analysis is to treat it as decomposition of the covariances among the tests in battery $\mathbf{t}$. In this case the object domain is the homogeneous set $\mathbf{t}$, the empirical relational system is { Cov(____ , ____) = c}, and the num relations which represent the latter are the vectorial inner products $\{P_c^2\}$ defined above. The difference between descriptive factoring and inductive factoring (i.e., factoring with observed variances in the diagonals of the empirical covariance matrix vs. factoring with communality-reduced diagonals) may be viewed as a contrast between no boundary restrictions (i.e., each B_c^2 is the universal property over $\mathbf{t}^2$) and the boundary restriction that the decomposition applies only to pairs of different tests.

Coombsian models. One of the most vigorous movements in recent psychometric research, inspired in large measure by the work of Clyde Coombs (1964), has been development of nonmetric models for analysis of empirical order comparisons. All of these are factorial decomposition models in which the observed ordering in an array of data objects (usually persons and/or stimuli) is hypothesized to correspond to the order of the distances separating these objects in an underlying m-dimensional genotypic space. For example, suppose that for each person j in population $\mathbf{j}$ and each pair of stimuli $\langle s_h, s_i \rangle$ from a stimulus array $\mathbf{s}$, we know empirically whether person j prefers s_h to s_i. Coombsian model for these comparisons maps each j in $\mathbf{j}$ and s in $\mathbf{s}$ into an m-component real-number vector, regarded as the genotypic coordinates of this object, such that j does or does not prefer s_h to s_t according to whether j's genotypic distance from s_h is or is not greater than j's genotypic distance from s_i; while 'distance' is determined by a function Δ which maps pairs of m-component number vectors into nonnegative reals. In the simplest models, Δ is euclidian—i.e., $\Delta(\langle x_{1a}, \ldots, x_{ma} \rangle, \langle x_{1b}, \ldots, x_{mb} \rangle) = \sqrt{\sum_{i=1}^{m} (x_{ia} - x_{ib})^2}$ —but Δ can also assume a noneuclidian form or be parametrically open.

Gravitational attraction. Given an array $\mathbf{o}$ of physical objects (chunks of minerals, etc.), it is possible to determine the force of attraction between any two of these objects as a function of the difference separating them. It is found that this force is inversely proportional to the square of the separation distance, but that the coefficient of proportionality is in general different for each pair of objects. The alternative possibilities for this attraction coefficient thus define a system of empirical relations over pairs of different objects from $\mathbf{o}$, the factorial decomposition of which is especially simple in that each o_i in $\mathbf{o}$ can be assigned a positive real number x_i such that for any two different objects o_i and o_j in $\mathbf{o}$, their attraction coefficient has value c iff $c = x_i \cdot x_j$.

Probabilistic dominance models. Suppose that for every two objects o_i and o_j in domain $\mathbf{o}$ there is a certain probability $\Pr(o_i \succ o_j)$ that o_i 'dominates' over o_j in some sense. For example, $\mathbf{o}$ might be a league of baseball teams while $\Pr(o_i \succ o_j)$ is the probability that team o_i will beat team o_j if they play; or $\mathbf{o}$ might be a set of auditory stimuli while $\Pr(o_i \succ o_j)$ is the probability that a randomly selected listener judges o_i to be louder than o_j. Then a mapping ψ of $\mathbf{o}$ into the real number domain and a function f over number pairs may exist such that for any two different objects o_i and o_j in $\mathbf{o}$, $\Pr(o_i \succ o_j) = f(\psi o_i, \psi o_j)$; while a hypothesis which implies that this possibility is, in fact, the case is a one-factor decomposition model for these dominance data. Several species of this generic model have appeared in the psychometric literature, notably Thurstone's 'law of comparative judgment' in which f is derived from the normal probability integral and the Bradley-Terry-Luce model in which $f(x, y) = x/(x+y)$ (see Luce, 1959a).

Models of risky choice. A large body of research on choice behavior has sought to analyze preference rankings among risky options as an outcome of cognitive and evaluative determinants. For example, let an agreement that person X will receive either payoff s_i or payof s_j according to whether or not uncertain event e turns out to occur to be called a 'dichotomous gamble' for X. Then if $\mathbf{e}$ is a set of uncertain events while $\mathbf{s}$ is a set of payoff states, each element of $\mathbf{e} \times \mathbf{s}^2$ is a possible dichotomous gamble for person X, and for every pair of such gambles it can be determined empirically which of the two X would prefer. A function ψ from $\mathbf{e} \cup \mathbf{s}$ into the real number domain such that person X prefers gamble $\langle e, s_i, s_j \rangle$ to gamble $\langle e', s'_i, s'_j \rangle$ iff $[\psi e \cdot \psi s_i + (1 - \psi e) \cdot \psi s_j] > [\psi e' \cdot \psi s'_i + (1 - \psi e') \cdot \psi s'_j]$ gives a factorial decomposition of X's preferences over this gamble domain. In a model of this sort (a 'subjective expected utility' model—see, e.g., Edwards, 1962), the values of ψ for events in $\mathbf{e}$ are thought of as the 'subjective probabilities' of these events, while ψ's values for payoffs in $\mathbf{s}$ are the 'utilities' which X attaches to these.

Genetic linkages. An especially interesting example of factorial decomposition is afforded by observable co-occurrences among inheritable traits, though in view of this pattern's complexity, I shall speak of it only with greatly oversimplified brevity. Suppose that $\mathbf{t}$ is an array of overt traits—barred eyes, vestigial wings, etc.—found in species S (historically, the fruit fly Drosophila). It is generally possible to determine for each pair of traits t_i and t_j in $\mathbf{t}$ an empirical 'linkage strength' index of how strongly t_i and t_j tend to co-occur in the interbred decendants of a cross between a member of S who displays both and one who shows neither. It is found that $\mathbf{t}$ can be partitioned into several groups such that any two traits in different groups have zero linkages, while the linkages within each group have a one-dimensional decomposition. That is, a mapping ψ of $\mathbf{t}$ into 2-component number vectors and a number-valued function f over pairs of the latter exist such that for any two traits t_i and t_j in $\mathbf{t}$, $f(\psi t_i, \psi t_j)$ equals the linkage strength between t_i and t_j while the value of f for any two arguments is a decreasing function of the difference between their second components so long as they have the same first component, and is zero otherwise. Modern genetics interprets the two components of ψt as, respectively, the chromosome-on-which-located and within-chromosome position of a gene governing trait t, while discovery and analysis of such empirical linkage patterns was historically one of the primary grounds for identifying 'genes' (which were initially but theoretical entities postulated to account for genetic inheritance) with physical substances at specific loci on microscopically observable structures within cell nuclei (Sturtevant, 1965)).

Simultaneous normalization of distributions. Research practice often desires its variables to be so scaled that their distributions in various populations have a common shape, usually normal; and if the data as originally scaled do not have this feature we may try to induce it by monotonic transformations. For example, given the joint distribution of number-valued variables ϕ and ψ in base population P, we might hope to find a monotonic rescaling $\phi' = f\phi$ of ϕ such that the contingent distribution of ϕ' at each value of ψ in P is normal. Since the existence of such an f is not mathematically guaranteed except in degenerate cases, data where simultaneous normalization is, in fact, possible generally show an empirical pattern which is inferentially provocative. Such patterning can be subsumed under factorial decomposition as follows: Let $\mathbf{P}$ be a set of populations within each of which is distributed a numerically scaled variable ϕ with range v, while Φ is the function whose value for a numerical argument x is the cumulative proportion of a normal distribution at a score x standard deviations above the distribution's mean. Then there exists a monotonic rescaling of ϕ whose distribution is normal within each population in $\mathbf{P}$ iff there is a function over $\mathbf{v} \cup \mathbf{P}$ which maps each v_i in $\mathbf{v}$ into a real number x_i and each P_j in $\mathbf{P}$ into a pair $\langle \mu_j, \sigma_j \rangle$ of real numbers such that ϕ's cumulative proportion at value v_i in population P_j is equal to $\Phi[(x_i - \mu_j)/\sigma_j]$.

Factorial Decomposition of Relational Variables The empirical relations decomposed in most of the preceding examples can be described most conveniently as the alternative values of a natural variable over a product-set domain. A variant of Def. 17 which applies more succinctly to such cases is:

DEF. 19. (a) A triple $\langle \psi, \{B_i^k\}, \{f_i^k\} \rangle$, is a factorial decomposition of a set $\{\phi_i^k\}$ of formal relational variables iff ψ is a function from an object domain $\mathbf{o}$ into some num-domain $\mathbf{n}$ while for each i in index set $\mathbf{i}$ and some $k \geq 1$, ϕ_i^k is a formal variable whose argument domain is a subset of product-domain $\mathbf{o}^k$, B_i^k is a k-adic relation over $\mathbf{o}$, and f_i^k is a function over $\mathbf{n}^k$ such that

$$(\forall o^k)[B_i^k(o^k) \supset \phi_i^k o^k = f_i^k \psi^* o^k].$$

(b) A quadruple $\langle \psi, \{\phi_i^k\}, \{B_i^k\}, \{f_i^k\}\rangle$ is a factorial decomposition of a set $\{\boldsymbol{\alpha}_i^{(k)}\}$ of natural relational variables iff, for each i in $\mathbf{i}$, ϕ_i^k is a formal scale for natural variable $\boldsymbol{\alpha}_i^{(k)}$ while $\langle \psi, \{b_i^k\}, \{f_i^k\}\rangle$ is a factorial decomposition of $\{\phi_i^k\}$.

That is, factorially decomposing a system $\{\phi_i^k\}$ of scaled relational variables consists in finding a mapping ψ of their arguments into nums, and a representation of each relational variable ϕ_i^k by a function f_i^k over k-tuples of nums, such that under boundary conditions B_i^k, ϕ_i^k analyzes as the product (i.e., composition) of f_i^k with ψ^*.

Factorial Decomposition *vs* Scaling

Comparison of Def. 17 with Def. 10 shows that scale interpretation is to factorial decomposition as species is to genus. For if $\langle \mathbf{B}, \mathbf{R}, \mathbf{Q}\rangle$ is an interpretation of formal scale $\phi = f\dot{\boldsymbol{\alpha}}$ for natural variable $\boldsymbol{\alpha}$, scaling transformation f is also the factorization function in decomposition $\langle f, \mathbf{B}, \mathbf{Q}\rangle$ of relations $\mathbf{R}$. Conversely, if a decomposition $\langle \psi, \mathbf{B}, \mathbf{Q}\rangle$ can be found for a system of $\boldsymbol{\alpha}$-value relations in which the factorization function is one-one, then $\psi\dot{\boldsymbol{\alpha}}$ is a scale for $\boldsymbol{\alpha}$ of which $\langle \mathbf{B}, \mathbf{R}, \mathbf{Q}\rangle$ is an interpretation.[13] (Still another relation between scale interpretation and factorial decomposition is that if ϕ is a scale for $\boldsymbol{\alpha}$ with range $\mathbf{n}$, while $[\mathbf{n}]$ is the set of unit classes of elements in $\mathbf{n}$ and $\mathbf{B}_0$ is the null set of boundary restrictions, then $\langle \phi, \mathbf{B}_0, [\mathbf{n}]\rangle$ is a factorial decomposition of monadic relations $\boldsymbol{\alpha}$.) Yet it is precisely those cases of factorial decomposition which are not scale interpretations which hold the most challenging potential for scientific discovery. I shall attempt to bring out the nature of this difference by drawing it as starkly as I can. Then, when the inferential significance of nonscaling decompositions is clear, we shall reexamine the scale-interpretation case to see whether it, too, may not at times share these inferential prospects in ways so far unrecognized by scaling theory.

To begin, whereas the 'objects' whose relations form the content of a factorial decomposition may in principle be values of a scientific variable, the objects in paradigmatic decompositions are part of the science's subject domain—i.e., are the

[13] Strictly speaking, descent from Def. 17 to Def. 10 also requires the further restriction that each R_i^k in $\mathbf{Q}$ is a relation only over the range of ψ. However, Def. 10's implication that a scale interpretation interprets only relations over the scale's range is a minor detail which has some technical importance (see, e.g., fn. 9) when relations are defined extensionally, but would be omitted from a nonextensional concept of scale interpretation.

arguments of the science's variables. Thus while scaling is a mapping into nums of certain properties (i.e., values of a natural variable) over a scientific subject domain, what are so mapped in a *typical* decomposition are members of the subject domain itself. Moreover, this mapping is many-one in the latter case whereas scaling is by definition one-one. The crucial distinction between these two is that while a one-one scaling function merely *stipulates* a correspondence between elements of a num array and the previously identified values of a certain scientific variable, the many-one mapping of a factorial decomposition *discloses* equivalence classes within the science's subject domain with respect to the decomposition's empirical content, and in so doing implicates a heretofore unrecognized nonrelational variable (or, when the factorization function is vector-valued, a set of nonrelational variables) in which the decomposed relations have their origin. Whereas to interpret a scale is merely to recognize that a previously known system of natural relations is isomorphic to a certain system of num relations, paradigmatic factorial decomposition *analyzes* natural relations to reveal that their patterning can be explained by certain hypothetical source variables whose num-scaled values for particular data objects are latent in the latter's relational properties. That is, once we discover that empirical relations $\mathbf{R}$ have factorial decomposition $\langle \psi, \mathbf{B}, \mathbf{Q} \rangle$ we find ourselves strongly inclined to infer from this the existence of a natural variable $\boldsymbol{\alpha}$ whose value is the same for two of its arguments iff the latter are mapped by ψ into the same num, while which relations in $\mathbf{R}$ hold for any k given data objects are determined by the latter's $\boldsymbol{\alpha}$-values (under boundary restrictions $\mathbf{B}$) in the fashion described by $\mathbf{Q}$. But while in paradigmatic decompositions our inference to the existence and nomic behavior of this variable is a genuine inductive leap, the $\boldsymbol{\alpha}$ 'inferred' in the special case of scale interpretation is simply the variable whose scale is being interpreted, the existence and $\mathbf{R}$-determinations of which were known at the outset.

For example, suppose that $W(o_i, O_j)$ is the relation 'o_i can usually beat o_j at Indian wrestling' over a domain of persons, that $\boldsymbol{\mu}$ is a natural muscle-strength variable which lawfully determines which of two persons can usually beat the other at Indian wrestling, and that this effect is represented on a numerical scale ϕ_μ for $\boldsymbol{\mu}$ by the relation 'is a larger number than' in such fashion that for any two different persons o_i and o_j, o_i can usually beat o_j at Indian wrestling iff $\phi_\mu o_i > \phi_\mu o_j$. Knowing these facts about $\boldsymbol{\mu}$, we can define the relation Sw ('is superior for Indian wrestling than') over values of $\boldsymbol{\mu}$ as

$$Sw(\mu_i, \mu_j) =_{\text{def}} (\forall o_h, o_k)[\mu_i(o_h) \cdot \mu_j(o_k) \supset W(o_h, o_k)]$$

and have $\langle \neq, Sw, > \rangle$ (where $\neq$ is the boundary restriction 'is not the same person as') as an interpretation of scale ϕ_μ in which is-a-larger-number-than represents the is-superior-for-Indian-wrestling-than relation over muscle-strengths. However, the corresponding factorial decomposition of Sw, namely $\langle f, \neq, > \rangle$ where f is the scaling function $\phi_\mu = f\dot{\boldsymbol{\mu}}$ simply gives us what we started with, i.e., a scaling of variable $\boldsymbol{\mu}$ and its nomic implications for Indian-wrestling superiority. On the other hand, suppose that we have no idea there is any such thing as muscle-strength, but have collected enough data from Indian-wrestling contests to conclude that relation W has a factorial decomposition $\langle \psi, \neq, > \rangle$. Then we have *discovered* the muscle-strength variable, scaled as ψ, within our W-data even though, insomuch as all we have thereby learned about ψ is its functional relevance for W, we will not yet be able to identify it *as* muscle-strength.

In short, the end product of a successful factorial decomposition is generally—excluding scale interpretations—a hypothesis about the theoretical origins of relational data and the laws by which the postulated source variables generate their observable consequences.[14] Since this inference follows with psychological immediacy albeit not deductive necessity from the observed data patterning, it is a form of reasoning which, together with others of its kind, may be described as *ontological induction* (Rozeboom, 1961).

There is a rather obscure philosophical thesis about the structure of reality, known as the 'doctrine of internal relations', which has importance for the overall reasonableness of factorial decomposition as a method of scientific inference. Since previous statements of this doctrine have not been altogether clear, I will forestall quibbling over my formulation of it by giving it a new title:

The Thesis of Relational Composition: Whenever a descriptive k-adic relation R^k holds for a k-tuple $\langle e_1, \ldots, e_k \rangle$ of entities, there exist nonrelational properties $P_1, \ldots, P_k$ possessed by $e_1, \ldots, e_k$, respectively, such that for any k-tuple $\langle x_1, \ldots, x_k \rangle$, $P_1(x_1)$ and ... and $P_k(x_k)$ jointly entail $R^k(x_1, \ldots, x_k)$.

That is, entities stand in relations to one another only as a consequence of their nonrelational attributes. While the Thesis obviously needs considerable sharpening and perhaps qualification, I believe it to be basically correct.[15] (Actually, the Thesis is trivially true under an extensional reading of its terms, and perhaps under certain nonextensional interpretations as well. When I say that I think it is basically correct, I mean correct in some *significant* sense.) If so, any relational

[14] That these variables must generally be theoretical, rather than definitional constructs out of the data variables, derives from the fact that in the less-than ideal decompositions of real data, the values of the factorization function computed for the data objects by, e.g., a least-squares or maximum-likelihood method, can be regarded only as an approximation to the true values of these objects on the source variables implicated by the decomposition.

[15] The only serious counterexamples I can think of, excluding such relations as Exemplification and Identity which are logical rather than descriptive, are spatiotemporal relationships. But while 'absolute' conceptions of space and time have long been held in low repute, I suggest that they would surely deserve reconsideration were spatio-temporal relations otherwise to be the Thesis's only exceptions.

data we may have are a resultant of the data objects' nonrelational attributes, and if we are not already cognizant of the latter, their disclosure should be the first target of our research in this area. If the Thesis is sound, therefore, factorial decomposition models are the methodologically preferred tools for analysis of relational data. On the other hand, it does *not* follow that *every* factorial decomposition furnishes inductive insight into underlying sources, for any system of relations has innumerably many trivial decompositions. Specifically,

THEOREM 7. If $\mathbf{R}$ is a set of relations over domain $\mathbf{o}$ while ψ is any one-one function with arguments in $\mathbf{o}$ and $\mathbf{B}_0$ is the null set of boundary restrictions, then $\langle \psi, \mathbf{B}_0, \psi^*\mathbf{R} \rangle$ is a factorial decomposition of $\mathbf{R}$.

For by Theorem 2, if we map object domain $\mathbf{o}$ into a num domain $\mathbf{n}$ by a one-one function ψ, any relation R_i^k over $\mathbf{o}$ is not only represented in $\mathbf{n}$ by its ψ-image $\psi^* R_i^k$ but is the interpretation of the latter in $\mathbf{o}$ as well.

The significance of Theorem 7 is not just that a factorial decomposition exists for every system of empirical relations—for the Thesis of Relational Composition says this should be so—but that the Theorem describes a *kind* of decomposition which always holds trivially irrespective of whatever genuinely interesting patterning the data may display. Two important facets of factorial decomposition are attested by this. The first is that a given relational system may sustain a plurality of decompositions which are independent in that one does not entail another. And secondly, it would be most imprudent to assume that every factorization function scales a variable which underlies the factored relationships in any scientifically meaningful sense. Each specific instance of factorial decomposition needs be examined in the fullness of its own detail before we can feel even modestly sure of what particular inference we should wish to draw from it; and how the differences among decomposition patterns modulate our intuitions about their scientific significance is still another important question awaiting study by the theory of data analysis. Most aspects of this problem lie far beyond our present scope, but two have immediate relevance for the shape of our thinking about scale interpretation.

If a factorial decomposition $\langle \psi, \mathbf{B}, \mathbf{Q} \rangle$ of empirical relations $\mathbf{R}$ over domain $\mathbf{o}$ is to disclose any new variables over $\mathbf{o}$, it is first and foremost necessary (with one partial exception noted later) that ψ be a many-one function, or at least that there be nothing about the decomposition which prohibits ψ's mapping different objects into the same num. For if ψ necessarily has a different value for each of its arguments, then these serve merely as num representations for the o's themselves, rather than for properties over $\mathbf{o}$. That is, unless the decomposition generates equivalence classes within the system's object domain, it merely states in isomorphic num terms what we knew to begin with, namely, which objects satisfy what $\mathbf{R}$-relations. Yet to be scientifically significant, the decomposition must also manage to do more than *merely* identify specific objects which are alike in respect ψ. For not only does the pursuit of scientific generality urge that the conclusions we draw be free of reference to particular objects, the number of admissible values for ψ may also be so large in comparison to the size of $\mathbf{o}$ (e.g., when $\mathbf{o}$ is finite while the theoretical variable manifested by ψ is continuous) that virtually all objects in $\mathbf{o}$ have unique ψ-values in fact even though there is no need for this in principle. But if factorial decomposition $\langle \psi, \mathbf{B}, \mathbf{Q} \rangle$ of $\mathbf{R}$ is to show more than the bare existence of underlying determinants, this something more must be conveyed, somehow, by the decomposition's *type*—i.e., by the nature of $\mathbf{R}$'s representation $\mathbf{Q}$. (To the extent that boundary restrictions $\mathbf{B}$ are nontrivial, they serve only to attenuate $\mathbf{Q}$'s import.)

But what hidden realities *can* a decomposition's type expose, anyway? On this point, unfortunately, my present insight offers little beyond one or two speculative possibilities and conviction that the problem is well worth sustained inquiry. My first speculation is that whatever it is in a decomposition's type which provokes substantive inferences, it is largely what governs our intuitive judgments about the 'strength' and 'simplicity' of data patterns. 'Strength' is what is at issue when we think of one decomposition as more demanding, more stringent, more tightly structured than another. Thus among decompositional mappings of data objects into m-component vectors, other things equal, the smaller is m the stronger is the pattern. (Somewhat more generally, I suspect that the intuitive 'strength' of a decomposition of relations $\mathbf{R}$ is closely related to how many units of $\mathbf{R}$-data are needed to determine the factorization function's value for a given object, or how rapidly an increasing number of such data converge upon this value.) As for 'simplicity', this is a concept which is still notoriously elusive, though paradigmatically, if the type $\mathbf{Q}$ of a factorial decomposition comprises polynomic functions, we feel that, other things equal, the lower the degrees of these polynomials the simpler is the pattern so represented. Even so, however decompositional 'strength' and 'simplicity' are to be explicated, we must not equate them too simply with the psychological convenience of the decomposition's type; for many different decomposition types, varying greatly in their mathematical tidiness, are entirely equivalent with respect to whatever inferences can be drawn from them:

THEOREM 8. If $\langle \psi, \mathbf{B}, \mathbf{Q} \rangle$ is a factorial decomposition of relational system $\mathbf{R}$ while f is a one-one function whose arguments include the domain of relations $\mathbf{Q}$, then $\langle f\psi, \mathbf{B}, f^*\mathbf{Q} \rangle$ is likewise a factorial decomposition of $\mathbf{R}$. *Corollary*: Relational system $\mathbf{R}$ has a factorial decomposition of type $\mathbf{Q}$ iff it also has a decomposition of type $f^*\mathbf{Q}$, where f is any one-one function over the domain of relations $\mathbf{R}$.

Let two factorial-decomposition types $\mathbf{Q}$ and $\mathbf{Q}'$ be said to belong to the same 'representation group' if there exists a one-one function f over the domain of $\mathbf{Q}$

such that $\mathbf{Q}' = f^*\mathbf{Q}$. Then Theorem 8 points out that a type-$\mathbf{Q}$ decomposition can always be transformed without cognitive loss into any other type in $\mathbf{Q}$'s representation group. It follows that whatever is scientifically significant about a factorial decomposition of types $\mathbf{Q}$ is not special to $\mathbf{Q}$ as such, but is common to all decomposition types in $\mathbf{Q}$'s representation group. In particular, either the factorial dimensionality or the polynomic complexity of a decomposition can to a large extent be selected at will (though not both at once) by an appropriate choice of the factorization function. The most insightful way for decomposition theory to proceed, therefore, should be through development of concepts which differentiate the structure of one representation group from another without special reference to any one particular member of the group, and then to ground our ideas of 'strength', 'simplicity', and whatever else in a factorial decomposition may be inductively significant upon these representation-group characteristics.

Of course, making clear the inferentially critical feature of decompositional structure will not in itself disclose what inferences these are critical *for*. One possibility for the latter will eventually surface in Part III, below. Meanwhile, it may be asked whether the general prospects for inductive revelation in factorial decompositions have any application at all to the special case of scale representation. We have already seen why the latter cannot generally be expected to tell anything about the variables scaled that was not known to begin with. Just the same, there are at least two ways in which a scale interpretation *might* be informative. One is that if we can find a particularly tidy num representation for content $\mathbf{R}$ of a scale $\phi = f\dot{\boldsymbol{\alpha}}$ for natural variable $\boldsymbol{\alpha}$ by allowing the values of ϕ to be m-component *vectors* even though $\boldsymbol{\alpha}$ was not defined vectorially to begin with, then we should suspect that $\boldsymbol{\alpha}$ may be the cartesian product of m distinct natural variables not heretofore recognized individually. (Note that if scaling transformation f is vector-valued, then f determines equivalence classes among the values of $\boldsymbol{\alpha}$ with respect to each component of f even though no two values of $\boldsymbol{\alpha}$ agree in all their f-components. The plausibility that $\boldsymbol{\alpha}$ is really a vector would be further enhanced were scale content $\mathbf{R}$ to include one or more relations R_i^k whose num representation Q_i^k is a cylinder set in some of its argument's components even though these components are not irrelevant for other relations in $\mathbf{R}$.) Secondly, to the extent that the structural properties of a factorial decomposition's type convey information about source variables over and above their bare existence, similar significance may well invest the types of scale interpretations as well. If so, study of what, substantively, is responsible for a natural variable's having scales of types belonging to one representation group rather than to another may furnish clues about the general significance of decompositional types. And insomuch as a scale's type is determined by its content, we are confronted once again with the question of whether anything methodologically useful can be said about what *kinds* of relations hold for the values of natural variables.

3. Scale Content and the Nature of Measurement

Consider the totality of k-adic relations over values of a natural variable $\boldsymbol{\alpha}$. What possible differences in kind are there among these? The question makes little sense if we construe them all to be nothing but subsets of $\boldsymbol{\alpha}^k$, but interesting distinctions appear when we lay aside this extensional pretense. Two, in particular, which I find instructive are an analytic/empirical contrast in the truth of an assertion '$R^k(\alpha^k)$' and a superordinate/coordinate distinction in its meaning.

Relation R^k is *analytic* in its arguments if the truth or falsity of '$R^k(\alpha_1, \ldots, \alpha_k)$' for k values of $\boldsymbol{\alpha}$ is inherent in their nature, and is *empirical* otherwise. For example, if α_1 and α_2 are the values 'wearing scarlet lipstick' and 'wearing pink lipstick', respectively, of the natural Lipstickcolor variable over a domain of women, the truth of 'α_1 is to wear a more intense color of lipstick than is α_2' follows analytically from the inherent saturation difference between scarlet and pink; whereas the truth of 'Jane Smith prefers α_1 to α_2' can only be determined empirically. The superordinate/coordinate distinction, on the other hand, concerns whether or not $\boldsymbol{\alpha}$-values are truly logical objects of the assertion '$R^k(\alpha_1, \ldots, \alpha_k)$'. Specifically, R^k is *coordinate* with its arguments if each predicate 'α_i' in '$R^k(\alpha_1, \ldots, \alpha_k)$' occurs *as* a predicate, i.e., with its argument place occupied by a logical constant or bound variable and its integration into '$R^k(\alpha_1, \ldots, \alpha_k)$' accomplished through a propositional operator (conjunction, negation, etc.); whereas if '$R^k(\alpha_1, \ldots, \alpha_k)$' is, or contains as component, a logically atomic proposition in which one or more of the 'α_i' occurs as a subject term, then R^k is *superordinate* to its arguments. For example, if $\boldsymbol{\alpha}$ is the Wearing-C-colored-lipstick variable, the dyadic statistic 'Women are kissed more often when α_i than when α_j' is coordinate with its $\boldsymbol{\alpha}$-value arguments, as is likewise the monadic relation 'Jane Smith is α_i'. In contrast, 'Most women prefer α_i to α_j is superordinate to the values of $\boldsymbol{\alpha}$, for while it, too, asserts a statistical relationship, it employs $\boldsymbol{\alpha}$-concepts irreducibly as subject terms rather than as predicates.

Given these two distinctions (whose tendency to blur under close scrutiny does not impair the use to be made of them here), we may say that the content of an interpretation $\langle \mathbf{B}, \mathbf{R}, \mathbf{Q} \rangle$ of scale ϕ for natural variable $\boldsymbol{\alpha}$ is analytic or empirical, and superordinate or coordinate, according to how the $\boldsymbol{\alpha}$-value relations in $\mathbf{R}$ stand in these respects. The analytic/empirical and superordinate/coordinate dichotomies thus generate a 2×2 classificational schema for possible scale contents, and I shall now offer a crude inventory of what more determinate kinds of content are to be found in these four categories.

The analytic-coordinate category can be reviewed quickly, for the simple reason that it appears to be empty. For if R^k is coordinate with its arguments, then the truth of '$R^k(\alpha_1, \ldots, \alpha_k)$' depends upon what objects instantiate $\alpha_1, \ldots, \alpha_k$, and this is always (?) an empirical matter. The varieties of empirical-coordinate scale contents, on the other hand, are more abundant; in fact, I have been able to spot three distinct sorts of these. The first consists of *instantiation citations* in which the properties which formally are the relation's arguments are attributed to specific objects—e.g.,

(a1) John Smith has value ____ of variable $\boldsymbol{\alpha}$. [A monadic relation over values of $\boldsymbol{\alpha}$.]

(a2) Mary Jones has been alternating between ____ and ____ this year. [A dyadic relation over values of the Wearing-C-colored lipstick variable.]

Assertions generated by sentence schemata such as these are altogether lacking in scientific generality, and I can think of no reason for scaling theory to heed them. Not so with the other two content varieties in this category, however. Both of the latter, which I shall call 'incidence rates' and 'nomic imports', respectively, comprise statistical attributes of a variable's values but differ markedly in their scientific significance. *Incidence rates* describe or compare the occurrence-frequencies of the properties to which they are ascribed, e.g.,

(b1) $p\%$ of population P have value ____ of variable $\boldsymbol{\alpha}$. [A monadic relation over values of $\boldsymbol{\alpha}$.]

(b2) The proportion of population P having either value ____ , ____ , or ____ of variable $\boldsymbol{\alpha}$ totals c. [A triadic relation over $\boldsymbol{\alpha}$-values.]

(b3) More members of population P have value ____ of $\boldsymbol{\alpha}$ than have value ____. [A dyadic relation over $\boldsymbol{\alpha}$-values.]

(b4) The proportion of population P having both the first component of value ____ of vectorial variable $\langle\boldsymbol{\alpha},\boldsymbol{\beta}\rangle$ and the second component of value _____ of $\langle\boldsymbol{\alpha},\boldsymbol{\beta}\rangle$ is c. [A dyadic relation over values of vectorial variable $\langle\boldsymbol{\alpha},\boldsymbol{\beta}\rangle$. This example is clumsy in words, but shows how the joint distributions of natural variables can be treated as scale content.]

In contrast, *nomic imports* describe how their arguments lawfully make a difference for what other properties are also possessed by objects which instantiate these arguments—e.g.,

(c1) β_i is the most frequent value of natural variable $\boldsymbol{\beta}$ among members of population P who have value ____ of variable $\boldsymbol{\alpha}$. [A monadic relation over $\boldsymbol{\alpha}$-values.]

(c2) The arithmetic mean of number-valued formal variable ϕ is greater among members of P who have value ____ of variable $\boldsymbol{\alpha}$ than among those who have value ____ . [A dyadic relation over $\boldsymbol{\alpha}$-values. Formal variable ϕ is here assumed to scale some natural variable other than $\boldsymbol{\alpha}$.]

(c3) Any solid object whose density is ____ will float on any liquid whose density is ____ . [A dyadic relation over densities.]

(c4) δ is the contingent distribution of variable ϕ among members of P who have the first, second, and third components, respectively, of values ____ , ____ , and ____ of vectorial variable $\langle\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}\rangle$. [A triadic relation over values of a vectorial variable. This relation—or more precisely the functional derived from it by letting parameter δ vary appropriately for each argument triple—describes how variable ϕ is jointly dependent upon variables $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ in population P.]

Incidence rates can often be given simultaneous scale representation by num relations with precisely the same degree of psychological simplicity as the relations represented. For if each different value of natural variable $\boldsymbol{\alpha}$ has a different frequency of occurrence in population P, we can scale $\boldsymbol{\alpha}$ by the function which maps each $\boldsymbol{\alpha}$-value into its occurrence-frequency; whence any given numerical comparison holds for a given k-tuple of scale values iff this same comparison holds for the relative frequencies of the corresponding k-tuples of $\boldsymbol{\alpha}$-values. However, I can think of no practical utility for scale representations of incidence rates. Very much in contrast to this is the interest which attaches to the scaling of nomic imports. It is by no means the case that lawful interdependencies among scientific variables can always be given mathematically powerful number representations; but when this *can* be done, a great many important advantages accrue to it, only one of which is the increased potency with which this facilitates generalization from the relational patterning of limited data. Although scaling theory is only now awakening to the full potential of nomic imports as scale content—for this is what 'conjoint measurement' is all about—scaling practice by research scientists has long chosen its scales with an eye to the mathematical simplicity these confer upon the representations of natural laws. Still entirely unexplored by either scaling theory or scaling practice, however, is the tantalizing metatheoretical question of *why* some but not all variables have nomic imports amenable to simple numerical representation. It is altogether possible that the representation group of conjoint scales whose content includes lawful dependencies among the variables scaled may be significantly determined by what *sorts* of variables these are. If so, then if we can discover these principles we will also be in position to infer things about a variable's nature from the representation group of its nomic imports. In particular, this is one way—perhaps the only one—in which a factorial decomposition's type may be able to reveal more about underlying source variables than their bare existence. But to find such principles, we must first identify aspects of a variable's nature which might be relevant. And since to analyze a natural variable's 'nature' is primarily to say what its values are like, what *might* be learned about a variable from the character of its nomic imports is how it stands in such respects as appear in the analytic-superordinate cell of our present classification.

Before considering scale contents which are inherent in their arguments, however, let us first dispose of the empirical-superordinate category. The only entries I can find for this are cases where a variable's values are appraised, designated, or otherwise cognitively related to by sentient beings—e.g.,

(d1) Most persons prefer ____ to ____ ,

(d2) John Smith thinks that ____ is more similar to ____ than to ____ ,

(d3) The phrase used most commonly in English to speak of ____ contains more letters than does the most common English phrase for ____ ,

all of which will take the values of any natural variable $\boldsymbol{\alpha}$ as arguments. Relations such as these derive from relations of higher polyadicity over $\boldsymbol{\alpha}$-values and persons, and data of the latter kind are best processed by factorial decomposition. If the resultant factorial mapping of $\boldsymbol{\alpha}$-values is many-one, then the factorization function determines equivalence classes among the α_i which may correspond to significant features of these; whereas if the mapping is one-one and hence merely a scaling of $\boldsymbol{\alpha}$, a possibility still remains that something further will be revealed about $\boldsymbol{\alpha}$'s nature by the decomposition's type. But either way, the inferential conclusions we draw from analysis of appraisal data depend upon, rather than lay a basis for, our theory of decompositional significance.

Finally, then, we come to scale-content resources in the analytic-superordinate category. A relation of this sort may be described as an 'intrinsic' or 'inherent' character of the arguments for which it holds insomuch as it depends only on the latter's nature. For example,

(e1) ____ is darker than ____ ,

(e2) ____ is more saturated than ____ ,

(e3) ____ has the same hue as ____ ,

are intrinsic dyadic relations over values of a natural Coloration variable over uniformly colored objects. While I can see no reason why there should not exist vast multitudes of inherent characteristics available for possession by the values of natural variables, I have been able to identify extremely few which are not of one special variety, and none at all outside this variety which have any generality beyond one or a few closely related variables. (E.g., what variables have values standing in saturation, brightness, or hue relations other than ones describable by predicate schemata of form 'having a P of color ____ ,' where P is some part or possession of the variable's argument?) Variables whose values have features of this special kind may appropriately be described as *quantitative*, and just what is meant by this is our last target for analysis.

Measurement: The Assessment of Quantity

Although this essay has throughout addressed matters commonly aired under the heading of 'measurement' theory, I have so far scrupulously avoided use of this term. It is now time to consider what measurement proper may be, and what it has to do with number assignment and scale content. While I have no desire to itemize all twists of meaning this word has sustained in its manifold applications (cf. the 40 variants cited by Lorge, 1951), I shall begin by noting distinguishable *themes* within the intuitive notion of 'measurement' in virtue of which this term is often extended beyond what, I shall argue, is its core sense.

The everyday grammar of 'measurement' employs this concept both as a verb and a common noun. The verb is transitive, occurring primarily in the linguistic context 'x measures Y', wherein x is a person and Y, the object of measurement, will be examined shortly. The noun's primary context, on the other hand, is 'X is a measure of Y', which still construes measurement as a dyadic relation but whose subject, X, is something other than a person.[16] In both its verb and noun forms, the concept's focus *qua* relationship is upon assessment: When person x measures Y, x *learns* (or attempts to learn) something about Y; while if X is a measure of Y, X makes available the sort of information which a person measuring Y seeks to acquire.[17] Clearly not every act of ascertainment counts as 'measurement', however, so the heft of this concept must lie in what sorts of entities are measured and what it is about them that measurement determines.

If such statements as

(*i*) The Wechsler-Bellevue is a measure of intelligence,

(*ii*) The Kuder Preference Record measures vocational interests.

(*iii*) The Standard Deviation and Probable Error are both measures of variation,

(*iv*) I'll have to measure that window's size before ordering a new pane for it.

(*v*) The first thing that cardiologist Jones does with a new patient is to measure his blood pressure and heart rate,

[16] 'X is a measure of Y' is also sometimes paraphrased transitively as 'X measures Y'. The latter derives from the former, however, and must not be confused with the *primary* verb form '(person) x measures Y' whose intransitive rephrasing ('person x is a measure of Y') yields only nonsense.

[17] With the notable exception of Leonard (1962), the modern measurement-theory literature is altogether devoid of any recognition that measurement is first and foremost an informational relationship. It could further be argued, perhaps, that 'x measures Y' also carries overtones of x's *doing* something to Y. But over and above the purely grammatical impact of the verb's transitivity, this is probably no more than a trace of the empirical fact that we usually need to interact with Y in order to assess it mensurationally.

(*vi*) Interferometer readings provide a more accurate measure of the speed of light than do direct distance-traveled observations.

(*vii*) The most important things in life, like health, love, and happiness, cannot be measured.

may be accepted as typical uses of measurement words, we can assert with considerable confidence that the grammatical object of measurement is either a scientific variable, as in examples (*i*)-(*iii*) and (*vii*), or, as in (*iv*)-(*vi*), a variable's value for a particular argument. Of these two, the latter appears primary—i.e., 'X is a measure of variable Y' seems elliptical for saying that on various occasions, X-data inform about specific values of Y. And what kind of information do they provide? Well, whatever else may be involved, to measure entity e's value of Y is first and foremost to identify *what* value of Y holds for e. This is, in fact, a theme so central to the intuitive concept of 'measurement' that often it appears to be essentially all that is meant. That is,

USAGE 1. In an extended sense of 'measurement', to measure entity e's Y (or, as it is sometimes put, to measure e with respect to Y) is to ascertain which, out of the alternative possibilities denoted by concept 'Y', is the particular one which obtains in e's case. In this same sense, variable X is a measure of Y if an entity's value of X reveals which Y-possibility it realizes.

There are, however, two importantly different *ways* in which status with respect to Y can be conveyed by values of X. On the one hand, as illustrated by examples (*i*), (*ii*) and (*vi*), X and Y may be two distinct variables whose correlational agreement permits an entity e's value of Y to be *inferred* from e's X-value. Thus,

USAGE 1a. In an extended sense, variable X is a 'measure' of Y under conditions C if X and Y are logically distinct variables whose joint distribution under conditions C makes X predictively relevant to Y.[18]

But a variable X may also be an *explication* of what it measures. As every professional scientist (if not every philosopher) is aware, the conceptual resources of ordinary language are profoundly inadequate for precise thinking about most matters; and where this deficiency is most acute is in describing particular values of variables. Virtually all commonsense adjectives do double duty for reference to (*a*) a poorly defined variable and (*b*) an even more poorly delimited region of values on that variable, while designation of specific alternatives under the variable is hopelessly beyond the capability of ordinary language. Consider, for example,

How tall is John? *Really* tall.

[18]For the technical details of such inferences, see e.g., Rozeboom (1966).

Jimmy's face is awfully dirty.

Jane is sexy, but not as sexy as her sister.

Jerry and Janice are both generous [brave, moody, friendly, honest, selfish, etc.], but are so in different ways.

It is clear in these statements that tallness, dirtiness, sexiness, generosity, etc., are fundamentally variables, only specific values of which hold for particular persons even though the corresponding adjectives 'tall', 'dirty', 'sexy', etc., mean something like 'having a value of the Tallness [Dirtiness, Sexiness, etc.] variable in the upper range of possibilities for this'. (This ambiguity is undoubtedly responsible for the increasingly prevalent use of the term 'attribute' in scientific writings to denote attribute *dimensions*, i.e., variables.) But while ordinary language has by now also acquired the technical ability to say precisely how tall John is (e.g., '76.8 inches'), it is quite incapable of putting into words (much less of ascertaining) *what* the different sexinesses are in virtue of which Jane is more so than her sister, or what specific attributes of the facial dirtiness or generosity sort a person might conceivably possess. When Y is a commonsense variable of this kind, identifying specific alternatives under the Y-idea is no small achievement, even though the technical variable X so defined, just because it *is* more precise than the commonsense notion of Y, is not altogether equivalent to the latter; and answering questions of how entities stand in respect Y by identifying their values on X is a second sense in which 'measurement' conveys information about what is measured. That is,

Usage 1b. In an extended sense, variable X is a 'measure' of Y when the alternative possibilities envisioned by term 'Y' are vaguely conceived while the values of X are an array of precisely defined, individually specifiable properties which intuitively qualify as being of kind Y.

Usage 1b is nicely illustrated by example (*iii*), above, which also points up the fact that the commonsense variable Y explicated by X may be multi-dimensional—i.e., X may be only one of many variables which are equally acceptable technical versions of Y. Usage 1b also clarifies the intent of disclaimers such as (*vii*). For when humanists deny that the things most dear to them can be 'measured', they are mainly incensed over the suggestion that a system of tidy abstractions can ever be devised to capture the rich, warm, infinitely nuanced particularity of these matters and serve them up immobilized, like butterflies pinned to a specimen board, for dissection and comparison—e.g., that the specific quality of our love or your happiness can ever be conveyed by impersonal scientific terms.

Despite our tendency to speak of 'measuring' a variable when our chief concern is only with precise determination of its values, however, there is more to measurement than just this. We would never consider, e.g., detection of a student's college major to be an act of measurement, nor would we likely admit that Interuterine-activity is a *measure* of Sex even if the sex of an unborn child could be diagnosed with high accuracy from its interuterine activity. Linguistic intuition insists that there is something *numerical* about measurement proper. Moreover, the latter is one theme which has suffered no neglect in the technical literature, for in one way or another virtually all modern measurement theorists have proposed that measurement is *numerical representation.* This tradition appears to have begun with Campbell's statement that "measurement is the process of assigning numbers to represent qualities" (Campbell, 1920, p. 267—though he goes on to argue that not any old numerical assignment counts as measurement), while Stevens has more recently and with fewer reservations expressed the same thought as "In its broadest sense measurement is the assignment of numerals to objects or events according to rules" (1951, p. 1). So widely has this view become disseminated that the descriptive semantics of 'measurement' must now recognize:

USAGE 2. In an extended sense, variable X is a 'measure' of variable Y if X is a number-valued scaling of Y.

But this is not really what measurement is. Brainsweat U.'s College-major-score scale does not become a *measure* of the natural College-major variable just by mapping college-major categories into numbers, nor would a Sex-scale variable on which the numbers one and zero represent 'male' and 'female', respectively, be happily regarded as a measure of sex. As we now conceive of them, College-major and Sex are just not the sort of variable which *can* be measured—not because we are unable to identify their specific values, but because they lack the character required of a variable which qualifies as measurable. And what is this missing character? The *word* for it is accessible enough, even if that word's meaning is rather more elusive. First note that one commonsensical reaction to the proposal that College-major and Sex can be measured is to reject this on grounds that the alternatives subsumed under these concepts differ in 'kind' rather than in 'degree'. But an intuitively equivalent (or nearly so) wording of this thought is to say that College-major and Sex are 'qualitative' rather than 'quantitative', whereas measuring consists in determining how much—i.e., *what quantity*—of something there is. I submit that unless a linguistically sensitive speaker of English is able to regard a thing as somehow 'quantitative', he would be loath to speak of *measuring* it without a concomitant display of 'as it were' signals. Thus,

USAGE 3. In the tough sense of the word, 'measurement' is *assessment of quantity.* Strictly speaking, person x measures entity e's Y iff x ascertains what quantity of Y is possessed by e; while variable X is a measure of Y iff X is a number-valued scale for Y on which a score n is the *amount* of Y-ness in the attribute whose X-scale value n is.

I shall not linger over arguments for the centrality of this usage, for I know of no instance where a writer who speaks of 'quantity' at all has not recognized its intimate relation to measurement, even though a contrast between 'intensive' and 'extensive' measurement or—what is not the same—between 'derived' and 'fundamental' measurement is occasionally drawn with the implication that only the second alternative in each contrast is truly an assessment of quantity. The real problem here is to say what it *is* for a variable to be 'quantitative'.

The literature on quantity and 'fundamental' measurement is so vast that a comprehensive review would be hopelessly impractical here even had I the fortitude to try. Instead, I shall summarize what has become the dominant view of this matter, show why it is unsatisfactory, and then sketch what in my judgment is the correct approach.

According to Campbell (1920) and most measurement theorists who have come after him, for a variable X over domain $\mathbf{d}$ to be 'fundamentally' measurable—i.e., for it to be quantitative—there must exist (a) a method for comparing $\mathbf{d}$-objects in virtue of which we can say that the X-value of one is inferior, equal, or superior to that of another, and (b) a physical concatenation operation which combines objects in $\mathbf{d}$ into aggregate $\mathbf{d}$-objects which likewise have X-values standing in this comparison relation. Then X is quantitative if these empirical relations can be represented on a number-valued scale for X by the order of numerical sums. The traditional example is Weight: Objects can be comparatively weighed by means of a balance, concatenation consists in lumping them together in the same balance pan, and it is found that weight can be so scaled that any concatenation of objects $d_h, \ldots, d_i$ outweighs the concatenation of objects $d_j, \ldots, d_k$ the scaled weights of the former have a larger sum than the scaled weights of the latter. This general idea can be expressed compactly, without an explicit ontology of aggregate objects, as follows: For every two integers j and k such that $1 \leq j < k$ let CX_j^k be the k-adic relation defined over $\mathbf{d}$ by object-concatenation and X-comparison operations in such fashion that $CX_j^k(d_1, \ldots, d_j; d_{j+1}, \ldots, d_k)$ holds for two groups of objects $d_1, \ldots, d_j$ and $d_{j+i}, \ldots, d_k$ iff the concatenation of objects in the first group is X-wise superior to the second-group concatenation. Then, says the Campbellian tradition, a function ω which maps each $\mathbf{d}$-object into a real number is a fundamental measure of X iff for each j-tuple $\langle d_1, \ldots, d_j \rangle$ and $(k\text{-}j)$-tuple $\langle d_{j+1}, \ldots, d_k \rangle$ of different objects in $\mathbf{d}$, $CX_j^k(d_1, \ldots, d_j, d_{j+1}, \ldots, d_k)$ iff $\sum\limits_{i=1}^{j} \omega d_i > \sum\limits_{i=j+1}^{k} \omega d_i$.

So described, Campbellian 'fundamental' measurement is actually a case of factorial decomposition. For it begins not with an array of alternative monadic properties over domain $\mathbf{d}$, but only with a relation CX over pairs of concatenated $\mathbf{d}$-objects. If, prior to the decomposition, we have already presumed the

CX-relation to manifest an underlying variable X over $\mathbf{d}$, then we shall also immediately construe the factorization function to be a measure of X; but until we are able to identify X's values independently of the CX-comparisons, X remains a hypothetical source variable *inferred* from the relational data. Even so, complicity of factorial decomposition is not the central point here. What is crucial to the Campbellian approach is that however the values of X are identified, they are taken to be *quantitative*—i.e., 'fundamentally' measurable—if and only if their causal implications for some binary comparison among physical concatenations of X's arguments have an additive scale representation.

But this seems to me to be altogether wrong. In the first place, it locates the essence of quantity in scale type rather than in scale content—specifically, in whether or not the variable in question has a scale which, relative to the causal outcome of certain combinatorial manipulations of the variable's arguments, is of type $\{\sum_{i=1}^{j} n_i > \sum_{i=j+1}^{k} n_i\}$. But what is so special about the latter that it should be definitive of 'quantity', especially when innumerably many other sets of numerical relations are representationally equivalent to it (e.g., $\{\prod_{i=1}^{j} n'_i > \prod_{i=j+1}^{k} n'_i\}$ into to which ithe summation inequalities are converted by rescaling-transformation $n' = 10^n$)? To reply—fairly enough so far as it goes—that what is essential here is not numerical additivity as such, but the abstract structure common to all relational systems in this representation group, still leaves mysterious why this particular structure should be singled out for special attention. More intolerable is that the Campbellian analysis seeks to find a variable's quantitativeness in its *nomic imports* rather than in its intrinsic character, and in a particularly restricted sort of effect at that.

Consider, for example, the quantitative variables over persons, Being x-inches tall and Having-y-siblings. We would agree, surely, that specific values of these variables, e.g., 'being 72 inches tall' and 'having three siblings', are quantities, and moreover that we can arrive at this judgment without first considering how they relate to their alternatives. Under the Campbellian conception of quantity, however, this intuitive judgment makes no sense insomuch as the quantitative character of being-72- inches-tall or having-three-siblings could appear only in the property's joint effect with others of its kind upon the comparative heights or number-of-siblings of person concatenations. Further, what would it *mean* to speak of such comparisons among concatenated persons? There are, to be sure, numerous ways in which two or more persons can be physically juxtaposed (acrobatic pyramids, subway crowds, sexual couplings, etc.), and some of these aggregates have greater vertical spread than do others. But even were there methods for conjoining people in such fashion that certain relations between these wholes are additively scalable outcomes of constituent-person heights, our belief that Height is quantitative cannot be based on such phenomena if only because we are not now aware of them. Whereas since only individual persons have siblings, it is not logically possible for the Number-of-siblings variable to be quantitative in virtue of its empirical effect upon concatenated siblinghood. Whatever we may be able to *infer* about a variable's quantitativeness from its nomic import for relations between aggregate arguments, the latter has no bearing upon the *meaning* of "quantity". When in paradigm cases we say that Height, Volume, Number-of-siblings, Number-of-hairs-on-head, etc., are quantitative while College-major, Lipstick-color and Sex are not, we are passing judgment not on what these variables *do* but on what they are *like*. The analysis of 'quantity' must thus be sought in terms of analytic-superordinate scale contents rather than in the empirical-coordinate terms of the Campbellian tradition.

But what *is* quantity, then? At risk of appearing hopelessly naive, I submit:

Principle of Quantitative Constitution: A value α of natural variable $\boldsymbol{\alpha}$ is a *quantity* iff α can be analyzed as a logical combination of a determinate number of distinct parts.[19]

For example, 'having three siblings' is the property of having a sibling, and another, and still another, and no more; while 'being 72 inches tall' consists in occupying a region of space which is analytically divisible into 72 horizontal strata, each an inch thick. To be sure, just what is to count as a 'logical combination of distinct parts' can well stand further illumination and I offer the Principle's present wording not as an ultimate insight but only as a guide to subsequent explication of specific quantitative concepts in their real-life employments in science and commerce. This is not a program to be realized in a few paragraphs; for not only are unambiguous examples of quantity much harder to come by than one might expect, the nature and manner of integration of a quantity's parts also show dubious consistency from one instance to another. (E.g., the way in which sibling-units compose a value of the Number-of-siblings variable does not, on the face of it, show much kinship to the way inch-units fuse into a value of Height-in-inches.) But even if the Principle is far from a last word as it now stands, I am convinced that it captures the essence of intuitive 'quantity'. There is, after all, nothing very original about it—the idea that quantities are aggregates of units is prominent not only in the commonsense concept but in the Campbellian analysis as well. The latter's error has been to regard 'units' as *objects* (i.e., as the bearers of properties) and their concatenation

[19] Further reflection has convinced me that α is better said to be a quantity iff α is the property of *having* an attribute analyzable into countable components of a certain kind, rather than being *itself* this attribute, so that to predicate quantity α of an argument a is to make an existence claim of form $(\exists P)[P(a) \cdot Q(P)]$ wherein Q details the relevant composition of P. However, this technical (though possible important) modification of the *Principle* does not essentially alter anything I say below. (Note added in proof.)

as a *causal* process unfolding in time; whereas understood properly, the units comprised by a quantity P are component *properties* of which P is an *analytic* aggregate.

Given a set $\mathbf{P}$ of properties, each member P_i of which is a constellation of countable components, any relation $Q^k(n_1, \ldots, n_k)$ over numbers defines a corresponding relation $Q^k(qP_1, \ldots, qP_k)$ over $\mathbf{P}$, where qP_i is the number of parts in property P_i. Often a quantity P_i can be analyzed into parts in more than one way, with the number of parts accordingly dependent on the choice of analysis. (E.g., the height which comprises 72 inch-units can just as well be conceptually partitioned into six foot-units, or into five foot-units plus 12 more inch-units.) In such cases, specification of quantity by number remains ambiguous until the kinds of units by which the quantity is parsed are also identified. Even so, any such analysis abstracts a genuine numerical aspect of P_i which stands in full-blooded numerical relations to the similarly abstracted numerical aspects of other quantitative properties. Thus 'being 72 inches tall' contains twice as many inch-units as does 'being 36 inches tall', while the number of siblings by which 'having three siblings' exceeds 'having one sibling' is equal to the number of sib-units by which 'having four siblings' exceeds 'having two siblings'. It is also impeccably true that 'being six feet tall' contains fewer foot-units than 'being 60 inches tall' contains inch-units, and that the number of inch-units in the latter is twenty times as large as the number of sib-units in 'having three siblings', even though we seldom find practical occasion to compare quantities articulated by units of different types.

I am perfectly aware that many of our quantity-comparison concepts involve more than just a counting of constituents, especially for quantities whose particulate composition is not perceptually or definitionally immediate but must be forcibly imposed by analytic contrivance as when, e.g., heights are dissected into components matching an arbitrarily chosen unit. Still insufficiently clear is what it means, e.g., to say that one height is n times as great as another even when no unit is specified, or to make comparisons among units of measurement themselves, or to speak of fractional and even irrational quantities. These are matters best reserved for another occasion, however, for they turn largely upon the explication of numerical concepts *per se*, a task which is still unfinished despite the important inroads by modern set-theoretical treatments of number. If the present rough sketch of quantity's lair does no more than revive awareness that this concept's true habitat is somewhere amid the formal complexities of compound predicates, rather than in the nomic-import bushes beaten by so many recent measurement theorists, its intent will be well served.

The Scientific Significance of Quantity

Outside observers of scientific practice have often commented (not always favorably) on the near-idolatrous esteem that quantity commands in science. In many cases the occasion for this judgment has undoubtedly been an ingenuous failure to distinguish numerical scaling from measurement proper; for while it is seldom that the variables studied in scientific research are not scaled numerically, it takes close scrutiny to tell whether this is anything more than a semantic convenience. Just the same, there are indeed good methodological reasons why true quantity is dear to the heart of technical science. How this is so will be our closing consideration; but first, we have some unfinished business to complete.

Regarding the relevance of quantity for scale content, it is obvious that if ϕ is a quantitative scale for natural variable $\boldsymbol{\alpha}$—i.e., if each ϕ-value is the number of u-units in the corresponding value of $\boldsymbol{\alpha}$—then any numerical relation Q^k over ϕ-values represents the quantitative $\boldsymbol{\alpha}$-value relation 'comprising u-units in respective amounts satisfying Q^k'. That is, numerical relations over quantitative scale values are essentially relations over the scaled quantities themselves, so that the distinction between scale type and scale content here virtually disappears. In particular, the 'additivity' which has so often been viewed as quantity's hallmark is merely a truism of arithmetic and as such has no more significance for measurement theory than does any other arithmetic property. (Thus if in my garage I have a pint of brake fluid, three quarts of distilled water, and a gallon of alcohol, then my garage contains 15 pints of liquid—not because these fluids would jointly fill a 15-pint container, but simply because mathematically, 1 unit + 6 units + 8 units = 15 units.) In short, when ϕ is a quantitative scale, we are free to interpret all numerical relations over ϕ-values in the commonsensical fashion which scaling theory rightly cautions against for numerical scales in general.

Our other unfinished business concerns the possible inferences to be drawn from scale representation of a variable's nomic imports. It is altogether possible for the values of natural variable $\boldsymbol{\alpha}$ to be quantities without our being aware that this is their nature, especially if $\boldsymbol{\alpha}$ is known to us only through factorial decomposition of relational data. Whether $\boldsymbol{\alpha}$ is quantitative, therefore, and if so, in what manner and of what units are its values composed, are questions which may conceivably be answered by the representation group of $\boldsymbol{\alpha}$'s lawful effects. For if the laws governing quantitative variables have a systematically different structure from laws in which nonquantitative variables participate, then, conversely, we should be able to infer something about a problematic variable's quantitativeness from the scale relations which are able to represent its nomic behavior. To be sure, whether there actually exist metaprinciples which relate a variable's character to the forms of the regularities which govern it as yet remains highly speculative—in fact, the bare suggestion is so unprecedented that many readers will likely reject it out of hand. Yet, surely it is not completely irrational to suspect that there are things to be learned by supplementing search for *what* natural regularities obtain with inquiry into *why* these are as they are, even if this will not be answered overnight or even in a generation. Meanwhile, a grain of encouragement perhaps lies in Stevens' (1957) empirical distinction between 'prothetic' and 'metathetic' sensory dimensions reflecting a possible distinction between how organic systems react to quantitative and qualitative inputs. Anticipation that quantitativeness affects nomic structure also has some rationalistic justification. For example, if the causal influences of quantities are exerted *per unit*, must this not impose some fashion of constraint on the resultant laws? And if further the quantity's articulation into units is a conceptual artifice which can be imposed in many different ways, might this not yield a concatenation of such constraints compatible only with certain restricted law-forms?[20] Whatever merit, or lack of it, these gropings may have, it remains a fact of human reason that we do at times infer quantity from nomic consequences, notably, in factorial decompositions of the Campbellian 'fundamental measurement' sort. For when C is a comparison relation whose arguments are physical aggregates of objects, while the latter have a mapping ψ into numbers such that a concatenation's C-comparison behavior is determined by the sum of its constituents' ψ-values, it is extraordinarily difficult to resist—nor should we necessarily resist—concluding that C-comparisons result from the juxtaposition of underlying quantities for which ψ is a quantitative scale. The psychological immediacy of this inference is precisely what makes it so easy to confuse the combinatorial *consequences* of quantity with its *nature*.

To conclude this prolegomenon to the theory of quantity, I shall suggest why it is that quantitative variables are so popular in scientific circles. Briefly, the point is that quantitativeness apparently provides the only route by which we can gain cognitive access to variables with a transfinite number of values. Before we can effectively study a natural variable $\boldsymbol{\alpha}$ we must possess linguistic resources for conceptualizing and communicating about each specific value of $\boldsymbol{\alpha}$. This does not demand that we ever make full use of this potential (an impossibility unless $\boldsymbol{\alpha}$ is finite), but it does require that we be able to identify any particular value of $\boldsymbol{\alpha}$ and to understand any such reference made by others. That is, for every value α_i of $\boldsymbol{\alpha}$, our language must contain a predicate a_i which designates α_i, at least in generative potential in the way, e.g., we have a name for every numerical integer even though no one could ever speak them all. But how can we acquire this array of concepts? If the number of $\boldsymbol{\alpha}$-values is finite, then each a_i can be logically primitive with its meaning imparted, say, by ostensive definition. But if $\boldsymbol{\alpha}$ has a transfinite number of values, then it is *not* possible for us to learn each a_i separately—instead, there

[20]What I dimly sense here is the possibility that Luce's Principle (Luce, 1959b), which is generically untenable (Rozeboom, 1962b), may in fact hold for certain kinds of quantitative relationships, in particular, those addressed by physicists in the theory of 'dimensional analysis'.

must be a generative pattern by which all but an at-most finite number of the a_i-predicates are built up through iteration of a learnably small number of meaning elements. (That an infinitude of $\boldsymbol{\alpha}$-values must all be different constructions out of the same atomic constituents seems requisite to our recognition of them as mutually exclusive.) The meaning content of such descriptive complexes can be of two basic kinds. On the one hand, a_i may specify intrinsic features of the property α_i it designates. In this case, the iteration of meaning elements in a_i may generally be regarded as a description of iterated units constituting α_i; whence α_i qualifies as a quantity. The other alternative is that a_i describes α_i in terms of its nomic correlates, notably, its connections with observation variables. For example, almost all variables studied by the more advanced sciences are measured by 'test' procedures wherein the outcome of an entity's standardized interaction with an assessment device is an index of its standing on an underlying variable $\boldsymbol{\alpha}$. (The theory of $\boldsymbol{\alpha}$'s relation to the test-outcome variable is usually grounded upon a factorial decomposition—often an intuitive one—in which $\boldsymbol{\alpha}$ is implicated by data of which test scores are but a small part.) In such a case, the instrument reading is only an imperfect causal consequence of the $\boldsymbol{\alpha}$-value so indicated, yet we are conceptually able to differentiate one α_i from another only *in terms of* the different test outcomes (and other observable effects) to which these give rise. Thus 'The value of $\boldsymbol{\alpha}$ under which τ_i is the most probable $\boldsymbol{\tau}$-test outcome' describes (in general, ignoring complications of nonmonotonicity) a different $\boldsymbol{\alpha}$-value for every different $\boldsymbol{\tau}$-alternative. But if a transfinite number of $\boldsymbol{\alpha}$-values are to be so identified, then $\boldsymbol{\tau}$ must likewise be transfinite—whence by our previous argument, if $\boldsymbol{\tau}$ is not extrinsically defined in terms of still other variables, it must in general be a quantitative variable. Thus, e.g., thermometer readings differentially identify a continuum of temperatures only because indicator-fluid volume is itself a continuous variable whose values are known to us quantitatively, and similarly for any measurement system in which the test outcome is literally a pointer reading.

The importance of quantity for identifying specific values of variables can also be appreciated through considering what character a natural variable must have in order that we be able to assign scale representations to its values. (E.g., a continuous ordinal variable cannot be scaled by reference to its ordinal features alone. Only if its values can be described metrically, either intrinsically or through the quantitative features of their effects, can we specify a particular mapping of them into numbers.) But this is an argument which can be saved for another occasion, for its main conclusion has already appeared: *With few if any significant exceptions, a transfinite-valued variable must either be intrinsically quantitative or have quantitative correlates in order for all its values to be individually conceivable by us.* Wherever nature is more abundantly diversified than finite categories will express, we can know its fine structure only in quantitative terms. Whether this is merely a Kantian coloration imposed by human limitation upon our image of the external world or corresponds to reality's ultimate texture, I do not care to venture.

References

Adams, E. W., Fagot, R. F., & Robinson, R. E. (1965). A theory of appropriate statistics. *Psychometrika*, *30*, 99–127.

Campbell, N. R. (1920). *Physics: The elements.* Cambridge: Cambridge University Press.

Coombs, C. H. (1964). *A theory of data.* New York: Wiley.

Edwards, W. (1962). Subjective probabilities inferred from decisions. *Psychological Review*, *69*, 109–35.

Leonard, H. S. (1962). *The use and abuse of measurement as a facet of scientific research.* Seattle: University of Washington Press.

Lorge, I. (1951). The fundamental nature of measurement. In E. F. Lindquist (Ed.), *Educational measurement.* Washington: American Council on Education.

Luce, R. D. (1959a). *Individual choice behavior: A theoretical analysis.* New York: John Wiley & Sons, Inc.

Luce, R. D. (1959b). On the possible psychophysical laws. *Psychological Review*, *66*, 81–95.

Luce, R. D., & Tukey, J. W. (1964). Simultaneous conjoint measurement: A new type of fundamental measurement. *Journal of Mathematical Psychology*, *1*, 1–27.

Rozeboom, W. W. (1961). Ontological induction and the logical typology of scientific variables. *Philosophy of Science*, *28*, 337-377.

Rozeboom, W. W. (1962a). Intentionality and existence. *Mind*, *71*, 15–32.

Rozeboom, W. W. (1962b). The untenability of Luce's principle. *Psychological Review*, *69*, 542–7.

Rozeboom, W. W. (1966). *Foundations of the theory of prediction.* Homewood, Illinois: The Dorsey Press.

Stevens, S. S. (1951). Mathematics, measurement, and psychophysics. In S. S. Stevens (Ed.), *Handbook of experimental psychology.* New York: John Wiley & Sons, Inc.

Stevens, S. S. (1957). On the psychophysical law. *Psychological Review*, *64*, 153–81.

Sturtevant, A. H. (1965). The fly room. *American Scientist*, *53*, 303–7.

Suppes, P., & Zinnes, J. L. (1963). Basic measurement theory. In R. D. Luce, R. R. Bush, & E. Galanter (Eds.), *Handbook of mathematical psychology* (Vol. 1). New York: Wiley & Sons, Inc.